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Discrete Mathematics 269 (2003) 249–258
DISCRETE MATHEMATICS
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The minimum number of vertices with girth 6 and degree set $D=\{r,m\}$ ☆

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Received 29 June 2001; received in revised form 28 March 2002; accepted 22 July 2002

Abstract

A $(D;g)$-cage is a graph having the minimum number of vertices, with degree set D and girth g. Denote by $f(D;g)$ the number of vertices in a $(D;g)$-cage. In this paper it is shown that $f(\{r,m\};6)\geqslant 2(rm-m+1)$ for any $2\leqslant r<m$, and $f(\{r,m\};6)=2(rm-m+1)$ if either (i) $2\leqslant r\leqslant 5$ and $r<m$ or (ii) $m-1$ is a prime power and $2\leqslant r<m$. Upon these results, it is conjectured that $f(\{r,m\};6)=2(rm-m+1)$ for any r with $2\leqslant r<m$.

Keywords: Cage; Girth; Degree set; Symmetric graph

1. Introduction

A (v,g)-cage is a graph having the minimum number of vertices, with valence v and girth g. The existence of (v,g)-cages was proved by Erdös and Sachs in the early of 1960s [4]. A $(D;g)$-cage is a graph which has the minimum number of vertices, with degree set D and girth g. It is obvious that the (v,g)-cage is a special case of the $(D;g)$-cage when $D=\{v\}$. Denote the number of vertices in the (v,g)-cage by $f(\{v\},g)$, which has the following property.

Lemma 1 (Longyear [7] and Wong [8]). *If* $k=v-1$ *is a prime power,* $f(\{v\},6)=2(k^2+k+1)$.

☆ The research done by the first author is partially supported by Chinese Natural Science Foundations.
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PII: S0012-365X(02)00758-6

The existence of $(D;g)$-cages has also been discussed in [1]. Denote the number of vertices in the $(D;g)$-cage by $f(D;g)$, which has the following properties.

Lemma 2 (Downs et al. [3]). *If $D=\{a_1,\dots,a_k\}$ with $2\leqslant a_1<\cdots<a_k$ and g is a positive integer with $g\geqslant 3$, then $f(D;g)\geqslant f_0(D;g)$, where*

$$f_0(D;g)=\begin{cases} 1+\sum_{i=1}^{t} a_k(a_1-1)^{i-1} & \text{if } g=2t+1,\\ 1+\sum_{i=1}^{t-1} a_k(a_1-1)^{i-1}+(a_1-1)^{t-1} & \text{if } g=2t.\end{cases}$$

Lemma 3 (Wong [7]). *For any $m\geqslant 3$ and $g\geqslant 3$,*

$$f(\{2,m\};g)=\begin{cases} \dfrac{m(g-2)+4}{2}, & \text{if } g \text{ is even},\\ \dfrac{m(g-1)+2}{2}, & \text{otherwise}.\end{cases}$$

Given a $(D;g)$-cage with degree set $D=\{r,m\}$ and girth $g\leqslant 5$, much effort has been taken in the past decades. For girths 3 and 4, Chartrand et al. [1] have shown $f(D;3)=1+a_k$ for $D=\{a_1,a_2,\dots,a_k\}$ and $f(\{r,m\};4)=r+m$ for any r with $2\leqslant r<m$. For girth 5, Downs et al. [3] have shown $f(\{3,m\};5)=3m+1$ for any $m\geqslant 4$, Limaye and Sarvate [5] have shown $f(\{4,m\};5)=4m+1$ for any even $m\geqslant 6$, we [8] have shown $f(\{4,m\};5)=4m+1$ for any integer $m\geqslant 5$, $f(\{5,m\};5)=5m+1$ for any $m\geqslant 6$. However, for the case where girth is 6, to the best of our knowledge, not much progress has been achieved.

In this paper, we deal with a $(D;g)$-cage where the degree set D is $\{r,m\}$ and the girth is 6. Our major contribution is to give a new lower bound for $f(\{r,m\};6)$, which is shown to be tight if either (i) $2\leqslant r\leqslant 5$ and $r<m$, or (ii) $m-1$ is a prime power and $2\leqslant r<m$.

The remainder of the paper is organized as follows. Section 2 provides the lower bound and upper bound for $f(\{r,m\};6)$, and Section 3 concludes the paper.

2. The bounds for $f(\{r,m\};6)$

Let u be a vertex in a graph, $d(u)$ be the degree of u and $N(u)$ be the set of neighboring vertices of u in the graph.

2.1. A lower bound for $f(\{r,m\};6)$

Following Lemma 2, it is easy to derive $f(\{r,m\};6)\geqslant 1+mr+(r-1)^2$. In the following we improve this lower bound by Theorem 1.

Theorem 1. *For any $2\leqslant r<m$, $f(\{r,m\};6)\geqslant 2(rm-m+1)$.*

Proof. Let $H_{r,m}$ be a graph with degree set $D=\{r,m\}$ and girth 6. For a given vertex $u\in V(H_{r,m})$ with $d(u)=m$, we distinguish its neighboring vertices into two cases: (1) there is a vertex $w\in N(u)$ with $d(w)=m$; (2) $d(w)=r$ for every vertex $w\in N(u)$. We deal with Case 1 first.

Case 1: There are two vertices $u_0, w_0\in V(H_{r,m})$ such that $d(u_0)=d(w_0)=m$ and u_0 is adjacent to w_0. Denote

$$N(u_0)=\{w_0,u_1,u_2,\ldots,u_{m-1}\},$$

$$N(w_0)=\{u_0,w_1,w_2,\ldots,w_{m-1}\}.$$

Since the girth of $H_{r,m}$ is 6, we have

$$N(u_i)\cap N(u_j)=\{u_0\},\quad 1\leqslant i<j\leqslant m-1,$$

$$N(w_i)\cap N(w_j)=\{w_0\},\quad 1\leqslant i<j\leqslant m-1,$$

$$N(u_i)\cap N(w_j)=\emptyset,\quad 1\leqslant i,j\leqslant m-1.$$

Therefore, we have

$$\begin{aligned}|V(H_{r,m})| &\geqslant 2+2(m-1)+\sum_{1\leqslant i\leqslant m-1}(d(u_i)-1)+\sum_{1\leqslant i\leqslant m-1}(d(w_i)-1)\\ &\geqslant 2+2(m-1)+2(m-1)(r-1)\\ &= 2(rm-m+1)+2(m-r)\\ &> 2(rm-m+1).\end{aligned}$$

We then proceed Case 2.

Case 2: For any given vertex $u\in V(H_{r,m})$ with $d(u)=m$ and $d(w)=r$ for all $w\in N(u)$. Let $v_0\in V(H_{r,m})$ be a vertex with $d(v_0)=m$. Denote

$$N(v_0)=\{v_1,v_2,\ldots,v_m\},$$

$$N(v_i)=\{v_0,v_{i,1},\ldots,v_{i,r-1}\},\quad 1\leqslant i\leqslant m,$$

$$N_v=\{v_0\}\cup N(v_0)\cup N(v_1)\cup\cdots\cup N(v_m),$$

$$N_u=V(H_{r,m})-N_v,$$

$$E_{vu}=\{e_{i,j}=(v_i,v_j):\ v_i\in N_v \text{ and } v_j\in N_u\}.$$

Then,

$$|N_v|=1+mr,$$

$$|E_{vu}|=\sum_{1\leqslant i\leqslant m,1\leqslant j\leqslant r-1}(d(v_{i,j})-1)\geqslant m(r-1)^2.$$

Consider the degree of a vertex $w\in N_u$; it can be classified into two subcases: either (2.1) $d(w)=r$ for all $w\in N_u$, or (2.2) there is a vertex w with $d(w)=m$.

Case 2.1: For any vertex $w \in N_u$, $d(w) = r$. Then,

$$|N_u| \geqslant m(r-1)^2/r = m(r-2) + m/r > 1 + m(r-2),$$

$$|V(H_{r,m})| = |N_v| + |N_u| > 1 + mr + 1 + m(r-2) = 2(rm - m + 1).$$

Case 2.2: There is a vertex $w \in N_u$ with $d(w) = m$. Denote

$$S_m = \{w_j:\ d(w_j) = m \text{ and } w_j \in N_u\},$$

$$|S_m| = s,$$

$$N(w_j) = \{w_{j,1}, w_{j,2}, \ldots, w_{j,m}\}, \quad 1 \leqslant j \leqslant s.$$

We have

$$|N(w_j) \cap N(v_i)| \leqslant 1, \quad 1 \leqslant j \leqslant s,\ 1 \leqslant i \leqslant m,$$

$$|N(w_j) \cap N_v| \leqslant m, \quad 1 \leqslant j \leqslant s.$$

Let

$$|N(w_j) \cap N_v| = y_j, \quad 1 \leqslant j \leqslant s,$$

$$y_t = \max\{y_j:\ 1 \leqslant j \leqslant s\},$$

$$u_0 = w_t, \quad y = y_t.$$

Denote

$$N(u_0) = \{u_1, u_2, \ldots, u_m\}.$$

Without loss of generality, we assume that

$$u_i = v_{i,1}, \quad 1 \leqslant i \leqslant y.$$

Denote

$$N(u_i) = \begin{cases} \{u_0, v_i, u_{i,1}, \ldots, u_{i,r-2}\} & \text{if } 1 \leqslant i \leqslant y, \\ \{u_0, u_{i,1}, \ldots, u_{i,r-1}\} & \text{if } y+1 \leqslant i \leqslant m. \end{cases}$$

We have

$$N(u_i) \cap N(u_j) = \{u_0\}, \quad 1 \leqslant i < j \leqslant y.$$

Let

$$x = |(N(u_{y+1}) \cup \cdots \cup N(u_m)) \cap (N(v_{y+1}) \cup \cdots \cup N(v_m))|.$$

Since the girth is 6, any of these x vertices and $u_1, \ldots, u_y$ do not have a common neighbor. Now, we consider the number of vertices in S_m. If $|S_m| = s = 1$ (see Fig. 1), then

$$x \leqslant (m-y)(r-1)$$

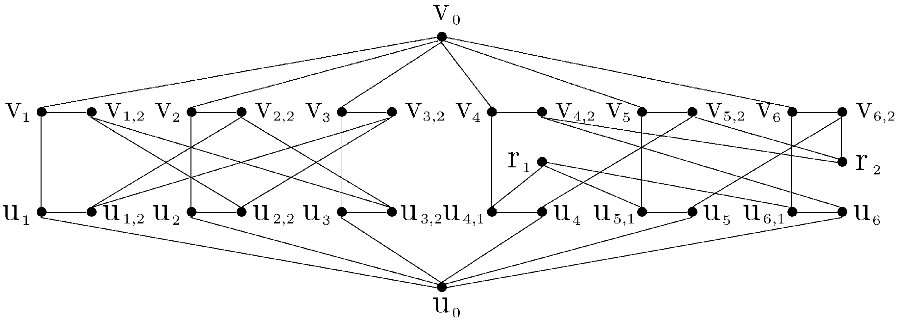

Fig. 1. $H_{3,6}$ with $y=3$ and $x=6$.

$$\begin{aligned}|N_u| &\geqslant 1+y(r-2)+(m-y)r-x+x(r-2)/r\\ &= 1+yr-2y+mr-yr-x+x-2x/r\\ &\geqslant 1+m(r-2)+2(m-y)-2(m-y)(r-1)/r\\ &= 1+m(r-2)+2(m-y)-2(m-y)+2(m-y)/r\\ &= 1+m(r-2)+2(m-y)/r\\ &\geqslant 1+m(r-2),\end{aligned}$$

$$|V(H_{r,m})|=|N_v|+|N_u|\geqslant 1+mr+1+m(r-2)=2(rm-m+1).$$

Otherwise ($|S_m|=s\geqslant 2$),

$$|N_u|\geqslant 1+m(r-2)+2s(m-y)/r$$

$$|V(H_{r,m})|=|N_v|+|N_u|\geqslant 1+mr+1+m(r-2)=2(rm-m+1). \qquad \square$$

2.2. An upper bound for $f(\{r,m\};6)$

We now consider the upper bound of $f(\{r,m\};6)$, which is stated by the following theorem.

Theorem 2. *If $k=m-1$ is a prime power with $2\leqslant r<m$, then $f(\{r,m\};6)\leqslant 2(rm-m+1)$.*

Proof. By Lemma 1, we have $f(\{m\},6)=2(k^2+k+1)=2(m^2-m+1)$ for a $(m,6)$-cage. Let H_m be an $(m,6)$-cage constructed in [7]. The $2(k^2+k+1)$ vertices in H_m are arranged as in Fig. 2. The set N_v consists of vertices $v_1,v_2,\ldots,v_k$, $v_{11},\ldots,v_{1k}$, $v_{21},\ldots,v_{2k},\ldots,v_{k1},\ldots,v_{kk}$. The set N_u can be defined similarly. Since k $(=m-1)$ is a prime power, there must exist a complete set of mutually orthogonal Latin squares $\{L_2,L_3,\ldots,L_k\}$ with elements $1,2,\ldots,k$ (see [2, p. 167, Theorem 5.2.4]).

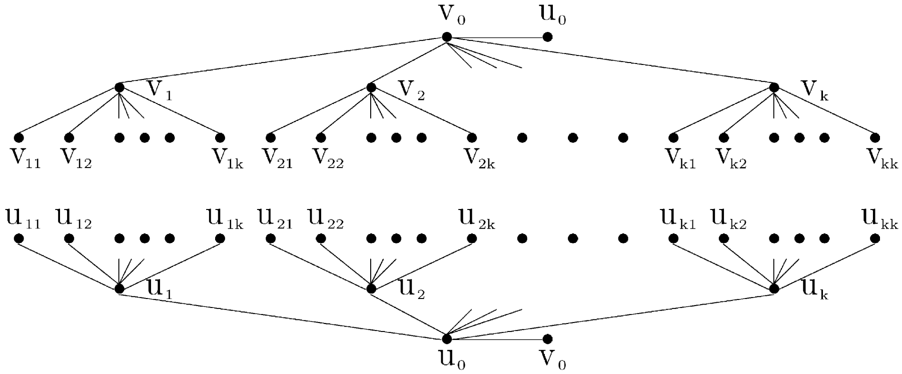

Fig. 2. An $(m,6)$-cage where $k(=m-1)$ is a prime power.

Let

$$L_1 = \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 1 & 2 & \dots & k \end{bmatrix}$$

and

$$L_t = \begin{bmatrix} L_{11}^t & L_{12}^t & \dots & L_{1k}^t \\ L_{21}^t & L_{22}^t & \dots & L_{2k}^t \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ L_{k1}^t & L_{k2}^t & \dots & L_{kk}^t \end{bmatrix}, \quad t = 2, 3, \dots, k,$$

where $L_{11}^t = 1, L_{12}^t = 2, \dots, L_{1k}^t = k$. The vertices of sets N_v and N_u are joined together according to the following rule:

$$v_{pq} \sim u_{1q}, u_{2L_{2q}^p}, \dots, u_{kL_{kq}^p} \quad (p, q = 1, 2, \dots, k),$$

where $a \sim b$ means that there is an edge in the graph between a and b. Since $L_2, L_3, \dots,$ L_k are mutually orthogonal Latin squares, it follows that H_m has girth 6 and valence m $(= k+1)$. For completeness, here we use an example to illustrate the construction.

Assume that $m=4$, then k $(=m-1=3)$ is a prime power. We have

$$L_1=\begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix},\quad L_2=\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix},\quad L_3=\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix}.$$

The edges in graph H_4 are given by the following table:

$$\begin{array}{lll}
v_{11}\sim u_{11} & v_{12}\sim u_{12} & v_{13}\sim u_{13}\\
\quad\quad u_{21} & \quad\quad u_{22} & \quad\quad u_{23}\\
\quad\quad u_{31} & \quad\quad u_{32} & \quad\quad u_{33}\\
v_{21}\sim u_{11} & v_{22}\sim u_{12} & v_{23}\sim u_{13}\\
\quad\quad u_{22} & \quad\quad u_{23} & \quad\quad u_{21}\\
\quad\quad u_{33} & \quad\quad u_{31} & \quad\quad u_{32}\\
v_{31}\sim u_{11} & v_{32}\sim u_{12} & v_{33}\sim u_{13}\\
\quad\quad u_{23} & \quad\quad u_{21} & \quad\quad u_{22}\\
\quad\quad u_{32} & \quad\quad u_{33} & \quad\quad u_{31}
\end{array}$$

Let $t=r-1$, $k=m-1$ and $G_{r,m}$ be the subgraph of H_m induced by the vertices in the set

$$\{v_0,v_1,v_{11},\ldots,v_{1k},v_2,v_{21},\ldots,v_{2k},\ldots,v_t,v_{t1},\ldots,v_{tk},$$
$$u_0,u_1,u_{11},\ldots,u_{1k},u_2,u_{21},\ldots,u_{2k},\ldots,u_t,u_{t1},\ldots,u_{tk}\}.$$

Following the above construction rules, the resulting graphs H_4 and $G_{3,4}$ for $r=3$ and $m=4$ are shown in Fig. 3.

Since $G_{r,m}$ has girth 6, degree set $D=\{r,m\}$, and $|V(G_{r,m})|=2(rm-m+1)$ vertices, we have $f(\{r,m\};6)\leqslant 2(rm-m+1)$ for a prime power $m-1$ and any r with $2\leqslant r<m$. The theorem then follows. □

We have already discussed the case where $m-1$ is a prime power. However, if $m-1$ is not a prime power, If is much harder to deal with. Here we only deal with the case for $r\leqslant 5$ through the construction of a $(D;6)$-cage with a degree set $D=\{r,m\}$, $r=3,4,5$ and $m>r$. We have the following theorem.

Theorem 3. *For $r=3,4,5$ and any $m>r$, $f(\{r,m\};6)\leqslant 2(rm-m+1)$.*

Proof. For $m=4,5,6$, it is obvious that $r=m-1$ is a prime power, and $f(\{r,m\};6)\leqslant 2(rm-m+1)$ by Theorem 3. For $r=3,4,5$ and $m\geqslant 7$, a graph $G_{r,m}$ is constructed as follows.

Denote by $e_{x,y}$ an edge in $G_{r,m}$ between vertices v_x and v_y, $0\leqslant x,y\leqslant 2(r-1)m+1$. Then,

$$V(G_{r,m})=\{v_0,v_1,\ldots,v_{2(r-1)m+1}\},$$

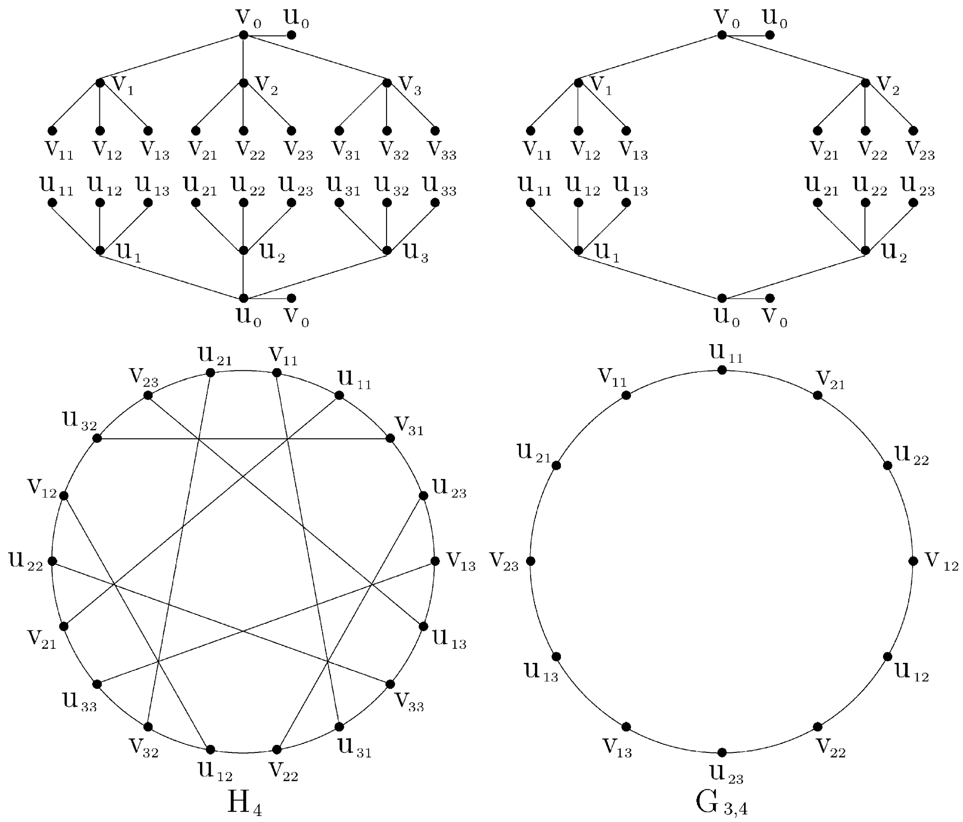

Fig. 3. An H_4 and $G_{3,4}$.

$$\begin{aligned}
E(G_{r,m}) = \{&e_{2(r-1)m+1,2(r-1)m+1-2i}, e_{2(r-1)m,2(r-1)m-2i},\\
&e_{2(r-1)m+1-2i,2(r-1)m-2i}\colon\ 1\leqslant i\leqslant m\}\\
&\cup\{e_{2(r-1)m+1-2i,(2m+1-2i+2mj)\bmod 2(r-2)m}\colon\ 1\leqslant i\leqslant m \text{ and } 0\leqslant j\leqslant r-3\}\\
&\cup\{e_{2(r-1)m-2i,(2m+4-2i+2mj)\bmod 2(r-2)m}\colon\ 1\leqslant i\leqslant m \text{ and } 0\leqslant j\leqslant r-3\}\\
&\cup\{e_{i,(i+1)\bmod 2(r-2)m}\colon\ 0\leqslant i\leqslant 2(r-2)m-1\}\\
&\cup\{e_{2i,(2i+5)\bmod 2(r-2)m}\colon\ 0\leqslant i\leqslant (r-2)m-1 \text{ and } r=4,5\}\\
&\cup\{e_{2i,(2i+3+2m)\bmod 2(r-2)m}\colon\ 0\leqslant i\leqslant (r-2)m-1 \text{ and } r=5\}.
\end{aligned}$$

Following the above construction rules, the resulting graphs $G_{4,7}$ and $G_{5,7}$ for $r=4,5$ and $m=7$ are shown in Fig. 4. Since graph $G_{r,m}$ has a degree set $D=\{r,m\}$, girth 6 and $|V(G_{r,m})|$ $(=2(rm-m+1))$ vertices, we have $f(\{r,m\};6)\leqslant 2(rm-m+1)$. The theorem then follows. □

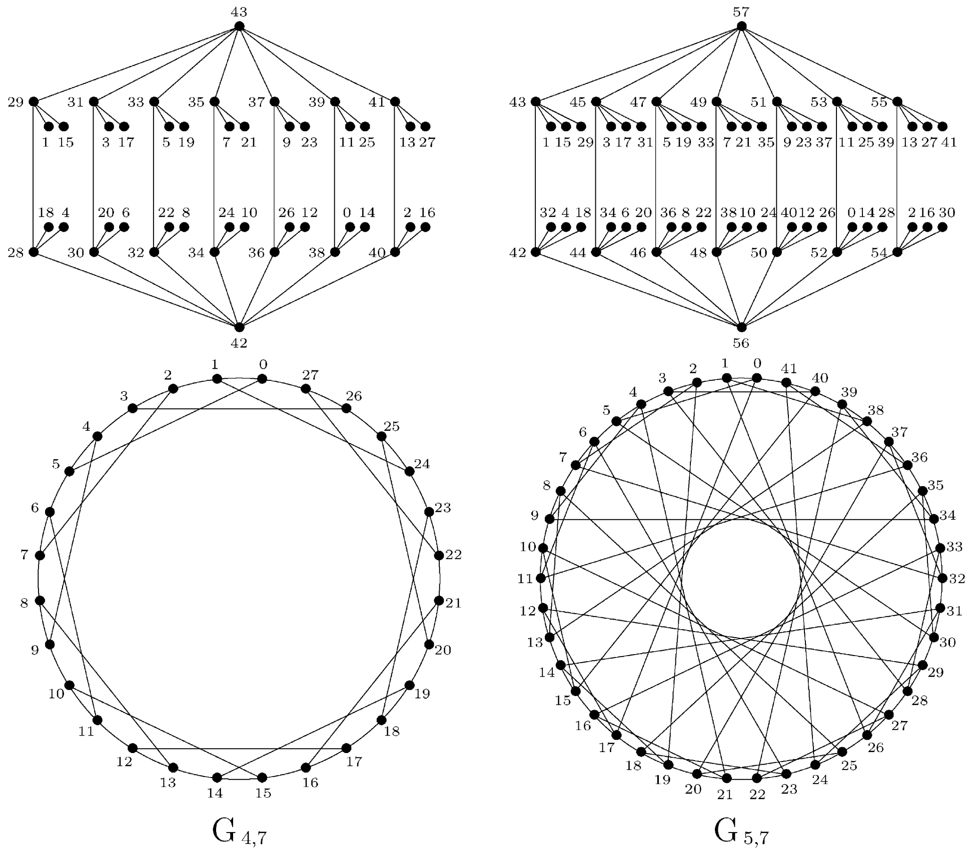

Fig. 4. An $G_{4,7}$ and $G_{5,7}$.

2.3. A tight bound for $f(\{r,m\};6)$

When $r=2$, $f(\{2,m\};6)=2m+2$ for any $m>2$ by Lemma 3. Following Theorems 1–3 and Lemma 3, we have

Theorem 4. *$f(\{r,m\};6)=2(rm-m+1)$, if either* (i) $2\leqslant r\leqslant 5$ *and* $r<m$, *or* (ii) $m-1$ *is a prime power and* $2\leqslant r<m$.

3. Conclusions

In this paper, we have shown that $f(\{r,m\};6)\geqslant 2(rm-m+1)$ for any r with $2\leqslant r<m$, and $f(\{r,m\};6)=2(rm-m+1)$ if either (i) $2\leqslant r\leqslant 5$ and $r<m$ or (ii) $m-1$ is a prime power and $2\leqslant r<m$. Upon these results, we have the following conjecture.

Conjecture. For any integer r with $2 \leqslant r < m$, $f(\{r,m\};6) = 2(rm - m + 1)$.

Acknowledgements

The authors are indebted to the anonymous referees for their constructive comments which helped to improve the presentation of the paper.

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