

Flow equivalent trees in undirected node-edge-capacitated planar graphs

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Abstract

Given an edge-capacitated undirected graph $G = (V, E, C)$ with edge capacity $c: E \mapsto R^+$, $n = |V|$, an $s - t$ edge cut C of G is a minimal subset of edges whose removal from G will separate s from t in the resulting graph, and the capacity sum of the edges in C is the cut value of C . A minimum $s - t$ edge cut is an $s - t$ edge cut with the minimum cut value among all $s - t$ edge cuts. A theorem given by Gomory and Hu states that there are only $n - 1$ distinct values among the $n(n - 1)/2$ minimum edge cuts in an edge-capacitated undirected graph G , and these distinct cuts can be compactly represented by a tree with the same node set as G , which is referred to the flow equivalent tree. In this paper we generalize their result to the node-edge cuts in a node-edge-capacitated undirected planar graph. We show that there is a flow equivalent tree for node-edge-capacitated undirected planar graphs, which represents the minimum node-edge cut for any pair of nodes in the graph through a novel transformation.

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1. Introduction

Given a node-edge-capacitated undirected graph $G = (V, E, c, u)$ with edge capacity $c: E \mapsto R^+$ and node capacity $u: V \mapsto R^+$, $n = |V|$ and $m = |E|$, an $s - t$ cut C of G is a minimal collection of nodes and edges whose removal from G will separate s from t in the resulting graph. An $s - t$ cut consisting of only the edges is referred to an $s - t$ edge cut. Clearly, $\{s\}$ and $\{t\}$ are trivial $s - t$ cuts. Any non-trivial $s - t$ cut is re-

ferred to an intermediate $s - t$ cut. The capacity sum of nodes and edges in an $s - t$ cut C is the cut value of C . A minimum $s - t$ cut is an $s - t$ cut with the minimum cut value among all $s - t$ cuts. A minimum $s - t$ edge cut is an $s - t$ edge cut with the minimum cut value among all $s - t$ edge cuts. A minimum intermediate cut is an intermediate $s - t$ cut with the minimum cut value among all intermediate $s - t$ cuts.

A theorem given by Gomory and Hu [4] says that there are only $n - 1$ distinct cut values among the $n(n - 1)/2$ minimum edge cuts in an edge-capacitated undirected graph $G(V, E, c)$. These cuts can be compactly represented by a tree T on the same node set of G such that the cut value of a minimum edge cut be-

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tween any pair of nodes in G is equal to the minimum edge capacity of the edges in the path between the pair of nodes in T . T thus is referred to *the flow equivalent tree* of G . Furthermore, the removal of any edge from a flow equivalent tree T separates the node set V into two disjoint subsets V_1 and V_2 , if the cut value of an $s - t$ edge cut of G is equal to the edge capacity of the removed edge in T , then T is referred to *the cut tree* of G , where $s \in V_1$ and $t \in V_2$. Both the flow equivalent tree and the cut tree can be constructed by performing $n - 1$ maximum flow computations in G , and the trees encode all the $n(n - 1)/2$ minimum edge cut values in an edge-capacitated undirected graph [4]. It is worth to mention that there is no such compact representation for all the $n(n - 1)$ minimum edge cuts in an edge-capacitated *directed* graph in which the edges are not symmetric, since there can be $(n + 2)(n - 1)/2$ different cut values of all minimum edge cuts [3].

In this paper we generalize the above problem further. That is, is there a flow equivalent tree as the compact representation of all cut values in a node-edge-capacitated undirected planar graph? There is a well known transformation, which transforms a cut between two specified nodes in a node-edge-capacitated undirected graph to an edge cut between the two corresponding nodes in an edge-capacitated directed graph, and the solution for the latter problem will return a solution for the original problem [1]. However, this transformation can not be used to construct a compact representation of all the cuts by a tree, because the symmetry of edge capacity is destroyed by the transformation [2]. Instead, we here show this is a flow equivalent tree for node-capacitated undirected planar graphs through a novel transformation, and the transformation maintains the symmetry of undirected graphs. To the best of our knowledge, this is the first solution for the problem in node-edge-capacitated undirected planar graphs.

2. Transforming a minimum intermediate cut to a minimum edge cut

In this section we show how to transform a minimum intermediate $s - t$ cut in a node-edge-capacitated undirected planar graph G between two nodes s and t to a minimum $s_e - t_e$ edge cut in an edge-capacitated undirected planar graph G_e between the two corresponding nodes s_e and t_e of s and t .

Given an undirected planar graph $G(V, E, c)$, denote by $G' = (V', E', l)$ the dual graph of G , where c is a function of edge capacity of edges in G and l is a function of edge length of edges in G' . Associated with each

edge $(i', j') \in E'$ in G' , there is a length $l(i', j')$, which is equal to the edge capacity $c(i, j)$ of its dual edge (i, j) in G . In terms of the relationship between G and G' , there is a well-known theorem as follows.

Theorem 1. [5] *The cut value of the minimum $s - t$ edge cut in G is equal to the minimum length of one cycle among all the cycles in G' that separate s from t in the plane.*

To transform a minimum intermediate $s - t$ cut in an node-edge-capacitated undirected planar graph $G(V, E, c, u)$ to a minimum $s - t$ edge cut in another edge-capacitated undirected planar graph G_e , the key is how to construct G_e . In the following G_e is constructed by replacing each node in G with a widget and linking all widgets together through the use of the corresponding edges of the edges in G . Specifically, given a node $v \in V$ of degree d in G , a widget of v is built as follows.

- (i) If $v = s$, or $v = t$, the widget consists of the node itself.
- (ii) If $d = 1$ and $v \neq s$ and $v \neq t$, the widget consists of only a single node v_1 .
- (iii) If $d = 2$, the widget consists of two nodes v_1 and v_2 , and an edge (v_1, v_2) . The capacity assigned to the edge is $u(v)$ —the node capacity of v in G .
- (iv) Otherwise, the widget is a circle consisting of d nodes v_1, v_2, \dots, v_d and d edges $(v_i, v_{(i+1) \bmod d})$, $1 \leq i \leq d$. The edge capacity of each of these edges is $u(v)/2$.

Having the widget ready, the rest is to link these widgets together by the corresponding edges of the edges in G , which is described as follows.

Given an edge $(v, w) \in E$, let d_v and d_w be the degrees of v and w in G , respectively. Assume that w_j and v_i are the corresponding nodes of w and v in the widgets of v and w , then, an edge (v_i, w_j) is added to G_e , and the edge capacity of the edge is equal to the edge capacity $c(v, w)$ of edge (v, w) in G , $1 \leq i \leq d_v$ and $1 \leq j \leq d_w$. Fig. 1(b) illustrates the construction of G_e , given the original graph G in Fig. 1(a).

We refer to G_e as *the $s - t$ extended graph* of G , and refer to the cycle in G_e consisting of nodes $v_1, v_2, \dots, v_d, v_1$ as *the chain-cycle* corresponding to node v in G . We thus have the following important lemma.

Lemma 1. *The cut value of a minimum intermediate $s - t$ cut in a node-edge-capacitated undirected planar*

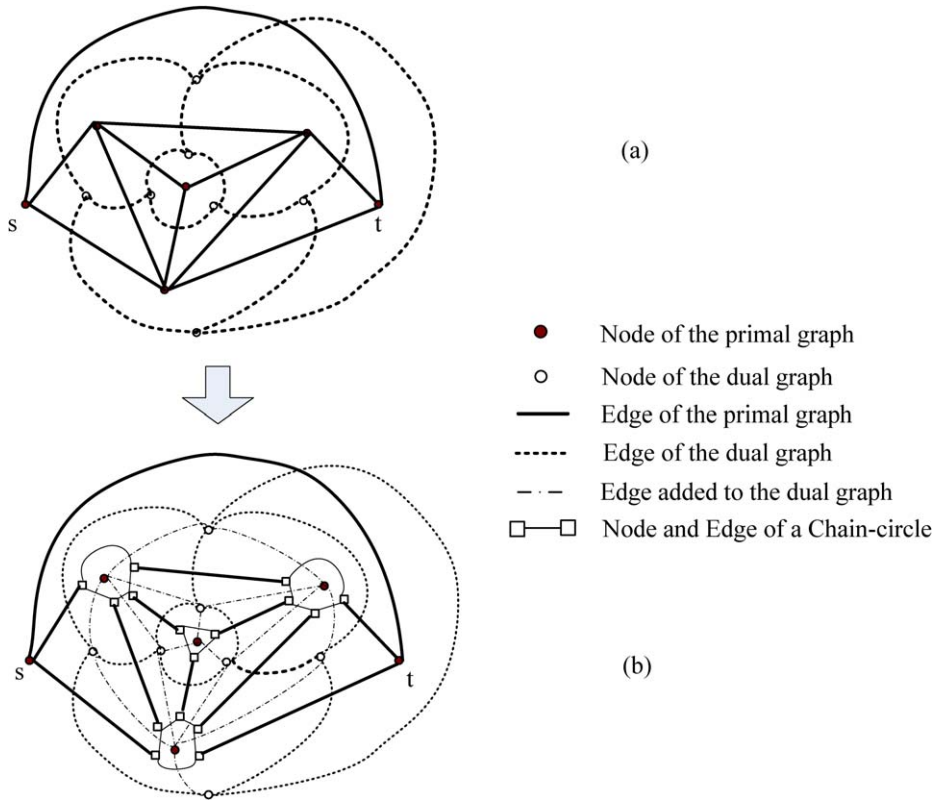


Fig. 1. (a) A primal graph and its dual graph. (b) The $s-t$ extended graph G_e and its extended dual graph G'_e .

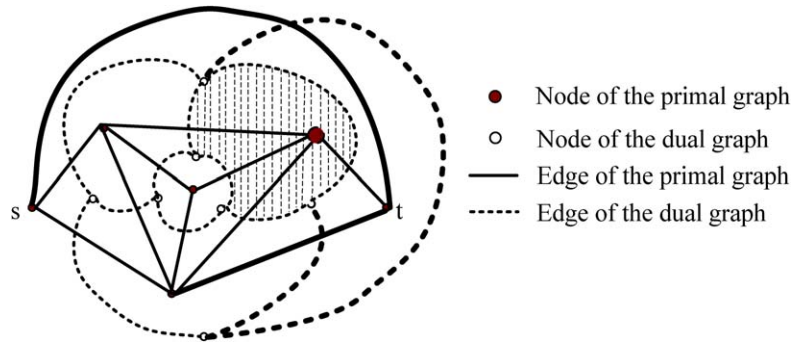


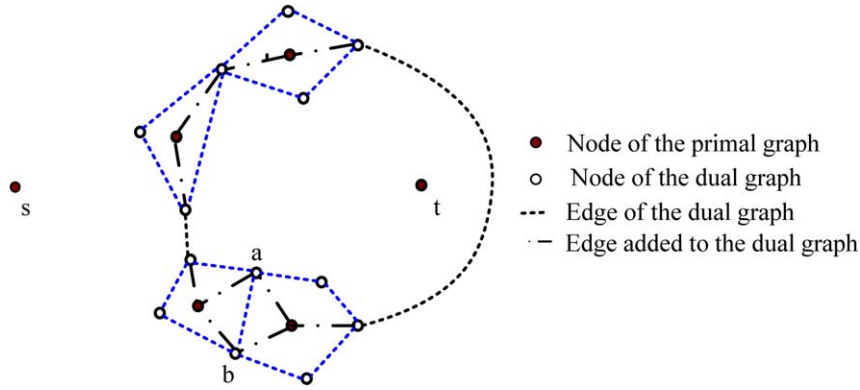
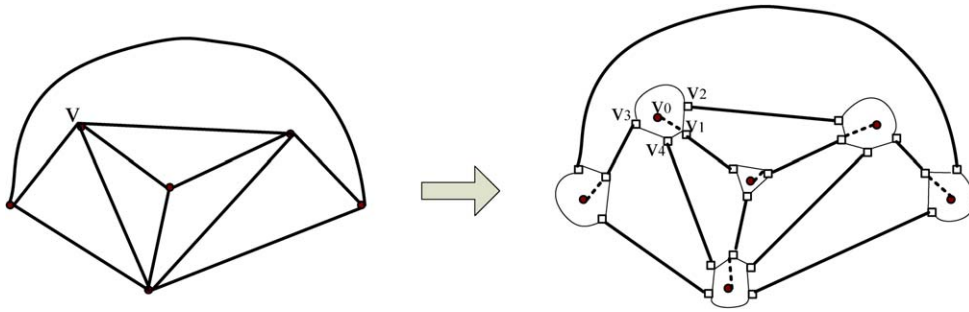
Fig. 2. A “cycle” in the dual graph G' of G corresponds to a node-edge cut in G , represented by the thick node and lines and the shade face.

graph G is equal to the cut value of a minimum $s-t$ edge cut in its corresponding edge-capacitated undirected planar graph G_e .

Proof. It can be seen that each intermediate $s-t$ cut in G corresponds to a “cycle” in its dual graph G' consisting of some dual edges and faces, which separates s from t in the plane (see Fig. 2).

To compute the cut value of the edge cut in G' , we extend G' to another graph G'_e as follows.

Add all other nodes in G except degree-one nodes, s and t to G' . For every newly added node v , add an edge between v and each node on the border of the corresponding face of v , and assign the edge a length that is equal to a half of the node capacity of v in G , i.e., $u(v)/2$. As a result, each “cycle” in G' that corresponds to an intermediate $s-t$ cut C in G contains one or more *derived cycles* in G'_e separating s from t in the plane (see Fig. 3). The minimum length of one derived cycle in G'_e among the derived cycles is equal to

Fig. 3. A “cycle” containing two cycles which are disjoint at nodes a and b .Fig. 4. Transform G to G^* .

the cut value of the intermediate $s - t$ cut C . Thus, the cut value of the minimum intermediate $s - t$ cut in G is equal to the minimum length of one derived cycle in G'_e among the derived cycles that separate s from t in the plane.

Following Theorem 1, we know that the minimum length of one derived cycle in G'_e among the derived cycles that separate s from t is equal to the cut value of the minimum $s - t$ edge cut in the dual graph G_e . This means that the cut value of the minimum intermediate $s - t$ cut in G is equal to the cut value of the minimum $s - t$ edge cut in the dual graph G_e of G'_e (see Fig. 1). \square

3. Constructing a flow equivalent tree

We now extend the transformation in the previous section to all pairs of nodes and deal with trivial cuts simultaneously. We transform a node-edge-capacitated undirected planar graph G to an edge-capacitated undirected planar graph G^* .

Recall that the planar graph G_e has already been built in previous section, given a planar graph G . The planar graph G^* is constructed as follows.

Add a new node v_0 and an edge (v_0, v_1) with edge capacity $c(v_0, v_1) = u(v)$. Fig. 4 illustrates the construction of G^* . As a result, there is a corresponding degree-one node v_0 in G^* for each node v in G , and node v_0 is marked to distinguish it from the other nodes in G^* . Note that the minimum $s - t$ cut is the one with the minimum cut value among two trivial cuts $\{s\}$ and $\{t\}$ and the minimum intermediate $s - t$ cut. By Lemma 1, it is not difficult to verify the following theorem.

Theorem 2. *The cut value of the minimum $s - t$ cut in G between a pair of nodes s and t is equal to the cut value of the minimum $s_0 - t_0$ edge cut in G^* , where s_0 and t_0 are the corresponding marked nodes of s and t in G^* .*

Thus, a flow equivalent tree for G can be built through the construction of a tree that represents the minimum edge cut value for any pair of marked nodes in G^* . The Gomory–Hu algorithm can simply be modified for this purpose. To do so, we adopt the notion of supernode. A *supernode* is a subset of nodes in graph G^* . Initially, the entire set of nodes in G^* is a supernode. The algorithm finds a minimum edge cut between any pair of marked nodes. The supernode is then further

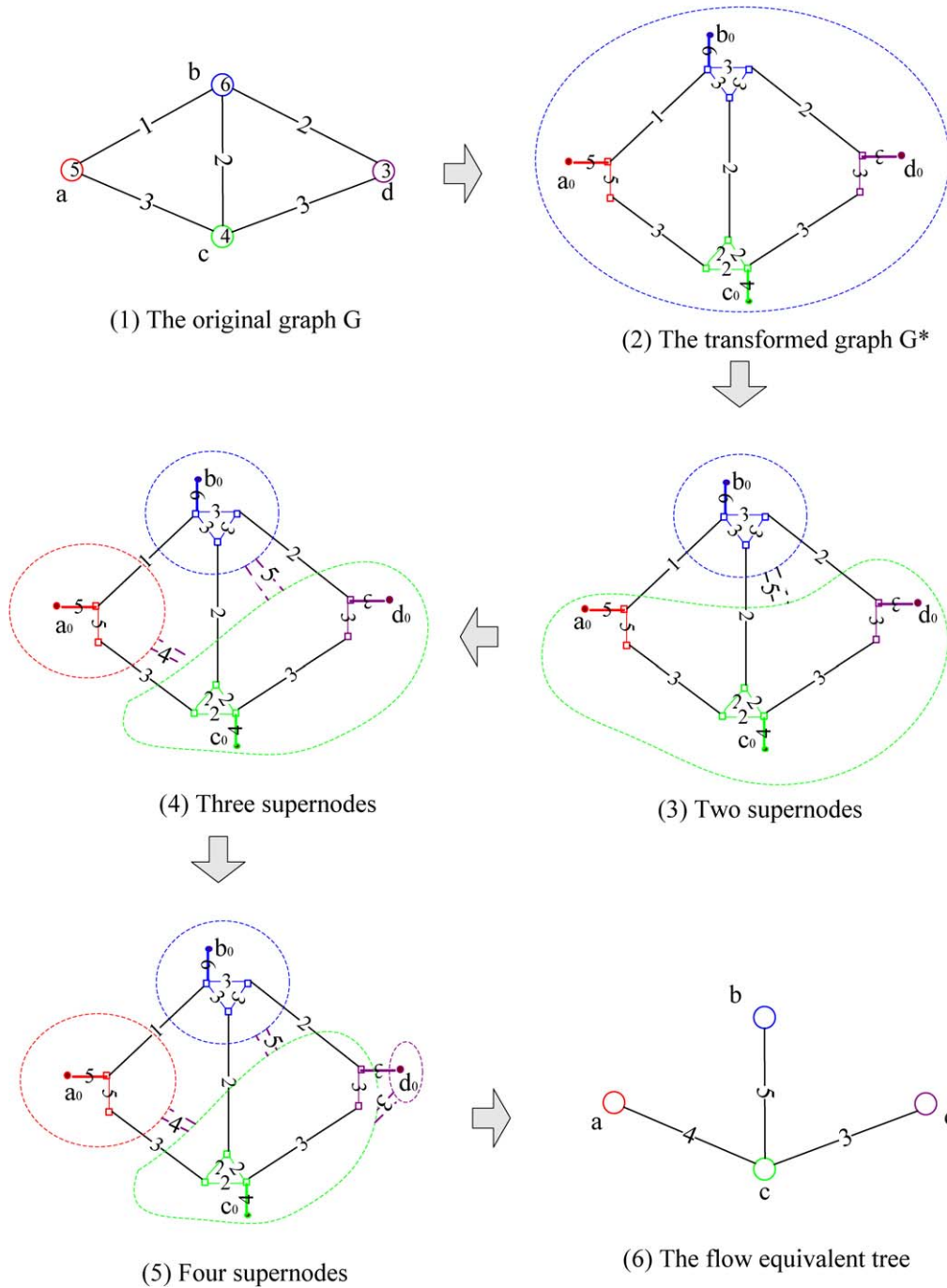


Fig. 5. The computation of flow equivalent trees.

partitioned into two supernodes by the cut, and the edge between the two supernodes with edge capacity being equal to the cut value. This procedure continues until every supernode contains only one marked node. In the following we use an example shown in Fig. 5 to explain the transformation. The original graph G is given

in Fig. 5(1), where the values are capacities of edges and nodes. Fig. 5(2) is the transformed graph G^* and its node set consists of only one supernode. Fig. 5 (3), (4) and (5) show how the supernodes are further partitioned. Fig. 5(6) is the resulting flow equivalent tree, where the value on each tree edge is the value of the

corresponding minimum cut between the two endpoints of the edge.

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