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Embedding *K*-ary Complete Trees into Hypercubes

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In this paper, dilated embedding and precise embedding of *K*-ary complete trees into hypercubes are studied. For dilated embedding, a nearly optimal algorithm is proposed which embeds a *K*-ary complete tree of height h, $T_K(h)$, into an $(h - 1)\lceil \log K \rceil + \lceil \log (K + 2) \rceil$-dimensional hypercube with dilation $\mathrm{Max}\{2, \phi(K), \phi(K + 2)\}$. $\phi(x) = \min\{\lambda: \sum_{i=0}^{\lambda} C_d^i \geq x$ and $d = \lceil \log x \rceil\}$. It is clear that $\lfloor (\lceil \log x \rceil + 1)/2 \rfloor \leq \phi(x) \leq \lfloor \log x \rfloor$, for $x \geq 3$.) For precise embedding, we show a $(K - 1)h + 1$-dimensional hypercube is large enough to contain $T_K(h)$ as its subgraph, $K \geq 3$.

I. INTRODUCTION

In recent years, many parallel algorithms have been designed to solve a variety of problems on various network topologies. Binary trees, meshes, and hypercubes are three important network topologies which have received extensive studies. Moreover, with the advance of VLSI, many new networks such as star graphs have been or will be proposed [13]. In order for existent algorithms on one architecture to be easily transformed to or implemented on another architecture, a new field deals with embedding among different networks [1]. Generally speaking, embedding is a mapping from the vertex set of one graph (called guest graph) to the vertex set of another graph (called host graph). Usually, four parameters, *dilation*, *expansion*, *load*, and *congestion*, are used to evaluate the performance of an embedding [5, 7]. An efficient simulation of one network on another network requires that these four parameters be as small as possible. However, for most embedding problems, it is almost impossible to obtain an embedding that minimizes these parameters simultaneously. Therefore, some tradeoffs among these parameters must be made [1, 2]. In this paper, we will discuss two kinds of embeddings: (1) *Precise Embedding*—an embedding with load one which maps two adjacent vertices in the guest graph G into two adjacent images in the host graph H; (ii) *Dilated Embedding*—an embedding with load one which allows two adjacent vertices in guest graph G to be mapped to distant images in host graph H.

The tree topology has a simple structure which is widely used in computer science. The hypercube topology has gained wide acceptance in parallel computing because it offers a high communication bandwidth, a small diameter, and a recursive structure naturally suited to divide-and-conquer applications [3]. Some research has been done on embedding special kinds of trees such as quadtrees or binary trees into hypercubes. The embedding of a complete binary tree into a hypercube was explored in [2, 4, 5], where it had been proven that any complete binary tree with height h cannot be precisely embedded into an $(h + 1)$-dimensional hypercube, even though the latter has more vertices than that of the former. However, a precise embedding of a double-rooted complete binary tree (DRCB) of height h, which is a variation of a binary tree, into an $(h + 1)$-dimensional hypercube has been obtained. This result implies a dilation 2 embedding of a binary tree with height h into an h-dimensional hypercube. Wu [5] presented an embedding of a K-ary tree with height h into an $(h - 1)\lceil \log K \rceil + 1$-dimensional hypercube. Compared with our dilated embedding, his embedding needs a slightly smaller dimension of the hypercube but needs dilation twice as large. Leiss and Reddy [6] presented a precise embedding of a complete binary tree of height h into an $(h + 2)$-dimensional hypercube which is an optimal embedding in terms of expansion. Ho and Johnson [7] proved that a complete quadtree of height h can be embedded into a $(2h + 1)$-dimensional hypercube within dilation 2, and Krishnakumar *et al.* [8] presented a specific algorithm for such an embedding. For arbitrary binary trees, Bhatt *et al.* [9] proved that any binary tree can be embedded into a hypercube by constant dilation and congestion. These constant bounds were later improved by Monien and Sudborough in [10]. Wagner [11] proved that any N-vertex binary tree is a subgraph of an $O(N \log N)$-vertex hypercube, and Wagner and Corneil [12] showed that the problem of determining whether a tree is a subgraph of a given hypercube is NP-complete.

In this paper, dilated embedding and precise embedding of K-ary ($K \geq 3$) complete trees into hypercubes are studied. A K-ary complete tree of height h, denoted $T_K(h)$, is a tree in which each internal vertex has exactly K children and the distance from the root to each leaf is exactly h. For each of these two embedding problems, we will (i) obtain a lower bound on the number of dimensions a hypercube must have so that $T_K(h)$ can be embedded; (ii) suggest an efficient algorithm to implement the embedding. Obviously, the K-ary complete tree is a more general model than the binary complete tree and the com-

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plete quadtree. It is expected that our research will provide some efficient tools for many simulation applications and a deeper understanding of these embedding problems.

The rest of this paper is organized as follows. In Section II the dilated embeddings are discussed. We will show lower bounds on expansions and dilations of embedding K-ary complete trees into hypercubes, and present an almost optimal embedding. In Section III we will obtain a lower bound on expansion of precise embeddings, and we will also present a precise embedding algorithm with a small expansion. In Section IV a summary on current results of embedding a complete tree into a hypercube is given.

II. DILATED EMBEDDINGS

2.1. Lower Bounds on the Expansions and Dilations

Let H_d be a d-dimensional hypercube. We assume that the vertices of H_d are labelled from 0 through $2^d - 1$ in binary such that two vertices are adjacent if and only if their binary labels differ in exactly one bit. We study the following question: what is the minimum dilation attainable by embedding $T_K(h)$ into its ideal hypercube, where the ideal hypercube is the smallest H_d which has at least $|V(T_K(h))|$ vertices?

Note that all embeddings concerned in this paper have load 1. Since $T_K(h)$ has a total of $(K^{h+1} - 1)/(K - 1)$ vertices, if $T_K(h)$ can be embedded into H_d, we must have $2^d \geq (K^{h+1} - 1)/(K - 1)$. Thus $d \geq \log((K^{h+1} - 1)/K - 1)) = \log(K^{h+1} - 1)\text{-}\log(K - 1)$. $d = \lceil \log(K^{h+1} - 1)\text{-}\log(K - 1)\rceil$ is the ideal dimension. It can be seen that $2^d > K^h$ and $d > \log K^h = h \log K$. Since d is an integer, we also have

$$d \geq \lceil h \log K \rceil. \tag{2.1}$$

By the definition, a lower bound of expansion for such embeddings is

$$(K - 1)2^{\lceil \log(K^{h+1}-1)\text{-}\log(K-1)\rceil}/(K^{h+1} - 1).$$

In order to obtain a lower bound on dilations, we need some properties of hypercubes. Given a vertex v in a d-dimensional hypercube H_d, let $\mathrm{Cs}(v, d, h)$ be the set of vertices including v whose distance to v is less than or equal to h; i.e., $\mathrm{Cs}(v, d, h) = \{u : u \in V(H_d)$ and $\mathrm{dist}(v, u) \leq h\}$, where $\mathrm{dist}(v, u)$ is the distance between v and u in the H_d.

It is easy to see that $|\mathrm{CS}(v, d, h)| = \sum_{i=0}^{h} C_d^i$. Given an H_d and an integer n, $0 \leq n < 2^d$, define the function $g(d, n)$ as

$$g(d, n) = \min \{\lambda: |\mathrm{Cs}(v, d, \lambda)| \geq n\}.$$

$g(d, n)$ is the smallest integer λ such that the number of vertices within distance λ from v in H_d is larger than or equal to n.

LEMMA 2.1. *Given a vertex v in an H_d, there exist n ($\leq 2^d - 1$) vertices, $v_1, v_2, \ldots, v_n$, such that* Max $\{\mathrm{dist}(v, v_i): 1 \leq i \leq n\} \leq g(d, n + 1)$.

Proof. Let $\lambda = g(d, n + 1)$; then $\mathrm{Cs}(v, d, \lambda) \geq n + 1$, i.e., $|\mathrm{Cs}(v, d, g(d, n + 1))| - 1 \geq n$. We can select $v_1, v_2, \ldots, v_n$ from $\mathrm{Cs}(v, d, g(d, n + 1)) - \{v\}$. ■

LEMMA 2.2. *Given a vertex v in an H_d, for any n distinct vertices, $v_1, v_2, \ldots, v_n, v_i \neq v$ $(1 \leq i \leq n)$, where $n \leq 2^d - 1$, we have*

$$\mathrm{Max}\, \{\mathrm{dist}(v, v_i): 1 \leq i \leq n\} \geq g(d, n + 1).$$

Proof. Because $|\{v, v_1, v_2, \ldots, v_n\}| = n + 1$, and $|\mathrm{Cs}(v, d, g(d, n + 1) - 1)| \leq n$, at least one vertex in $\{v_1, v_2, \ldots, v_n\}$ belongs to $V(H_d) - \mathrm{Cs}(v, d, g(d, n + 1) - 1)$, whose distance to v is larger than or equal to $g(d, n + 1)$. ■

For the special case $d = \lceil \log n \rceil$, we define a function $\phi(n) = g(\lceil \log n \rceil, n)$. The following inequality can easily be verified and provides bounds on $\phi(n)$:

$$\lfloor (\lceil \log n \rceil + 1)/2 \rfloor \leq \phi(n) \leq \lfloor \log n \rfloor. \tag{2.2}$$

LEMMA 2.3. *A lower bound on the dilation of embedding a $T_K(h)$ into its ideal H_d is $\phi(N)/h$, where $N = (K^{h+1} - 1)/(K - 1)$.*

Proof. Let f be a function that embeds a $T_K(h)$ into its ideal H_d. From the definition of function ϕ and Lemma 2.2, there must exist a nonroot vertex w in $T_K(h)$ such that $\mathrm{dist}(f(r), f(w)) \geq \phi(N)$ where r is the root of $T_K(h)$. Assume that the vertex sequence on the path from r to w in the $T_K(h)$ is $r = w_0, w_1, w_2, \ldots, w_s = w$. Then the image of this path P' in H_d is $f(r), f(w_1), f(w_2), \ldots, f(w_s)$ $(= f(w))$. Because

$$\begin{aligned}\phi(N) \leq \mathrm{dist}(f(r), f(w)) &\leq \mathrm{dist}(f(r), f(w_1)) \\ &+ \mathrm{dist}(f(w_1), f(w_2)) + \ldots \\ &+ \mathrm{dist}(f(w_i), f(w_{i+1})) + \ldots \\ &+ \mathrm{dist}(f(w_{s-1}), f(w_s)),\end{aligned}$$

and $\mathrm{dist}(r, w) = s \leq h$, we claim that there exists at least one i, $0 \leq i \leq s - 1$, such that

$$dis(f(w_i), f(w_{i+1})) \geq \phi(N)/s \geq \phi(N)/h,$$

i.e., the dilation of embedding f is at least $\phi(N)/h$. ■

Note that when $h > 1$, by Lemma 2.3, we have seen that a lower bound on the dilation of embedding $T_K(h)$ into its ideal hypercube is $\phi(N)/h$. By Eq. (2.2),

$\phi(N)/h \geq \lceil(\lceil\log N + 1)/2\rceil/h \approx \lceil(\lceil\log K\rceil + 1)/2\rceil$, and

$\phi(N)/h \leq \lceil\log((K^{h+1} - 1)/(K - 1))\rceil/h \approx \lceil\log K\rceil$.

Therefore, $\phi(N)/h$ and $\phi(K)$ are very close to each other.

2.2. An Almost Optimal Dilated Embedding

In this section, we will show how to embed a $T_K(h)$ into an $(h - 1)\lceil\log K\rceil + \lceil\log(K + 2)\rceil$-dimensional hypercube with dilation Max{2, $\phi(K)$, $\phi(K + 2)$}. From the above discussion, we know that the dilation of proposed embedding almost reaches its lower bound, and the expansion is slightly larger than its lower bound, since $(h - 1)\lceil\log K\rceil + \lceil\log(K + 2)\rceil$ is slightly larger than $\lceil h \log K\rceil$. Therefore, the proposed embedding is nearly optimal.

Because H_d is vertex and edge symmetrical, for any two vertices v_1 and v_2, there exists an automorphic mapping s from $V(H_d)$ to $V(H_d)$ such that $s(v_1) = v_2$. Also, given any two edges, $e_1 = (v_1, u_1)$ and $e_2 = (v_2, u_2)$, there exists an automorphic mapping s' from $V(H_d)$ to $V(H_d)$ such that $s'(v_1) = v_2$ and $s'(u_1) = u_2$. The embedding uses an inductive construction. Let r be the root of $T_K(h + 1)$, let $r_1, r_2, \ldots, r_K$ be the K children of r, and let $T_1, T_2, \ldots, T_K$ be the K K-ary complete subtrees of height h rooted at $r_1, r_2, \ldots, r_K$ respectively. Let s and s' be two automorphic mappings as defined above. If f is an embedding from $T_K(h)$ into H_d which maps r into v_1, then $s \circ f$ is an embedding from $T_K(h)$ into H_d which maps r into v_2. If f' is an embedding from $T_K(h)$ into H_d which maps edge (r, r_1) into edge (v_1, u_1), then $s' \circ f'$ is an embedding from $T_K(h)$ into H_d which maps (r, r_1) into (v_2, u_2). A vertex v in H_d is called *idle* under embedding f if no vertex in $T_K(h)$ is mapped into v by function f.

Let $d(h)$ be the minimum dimension such that $H_{d(h)}$ is large enough to embed $T_K(h)$ by our method (later we will show that $d(h) = (h - 1)\lceil\log K\rceil + \lceil\log(K + 2)\rceil$), and let H' be a $\lceil\log K\rceil + d(h)$-dimensional hypercube. Let $p = \lceil\log K\rceil$, $q = d(h)$, and the vertex set of H' be $V(H') = \{b_{p+q-1} \ldots b_{q+1}b_q \ldots b_1b_0 : b_i \in \{0, 1\}, 0 \leq i \leq p + q - 1\}$, where each vertex is labeled with a $(p + q)$-bit binary number. Now we construct a partition on $V(H')$: $V[0], V[1], \ldots, V[K' - 1]$, where $K' = 2^p$ and $V[j] = \{b_{p+q-1} \ldots b_{q+1}b_q \ldots b_1b_0 : b_{p+q-1} \ldots b_q$ is the binary representation of integer $j\}$, $0 \leq j \leq K' - 1$. Let $H[i]$ denote the q-dimensional subcube induced by $V[i]$. Thus, we call the set of K' subcubes: $\{H[0], H[1], \ldots, H[K' - 1]\}$ a K'-partition of H_d. A *representative hypercube* on the K'-partition, denoted H^*, is defined as a p-dimensional subcube induced by all vertices whose last q bits in their labels are fixed. There are $2^{d(h)}$ representative hypercubes for a given K'-partition. Thus, each vertex in H^* represents a unique $d(h)$-dimensional subcube in the K'-partition. On the other hand, each vertex in a $d(h)$-dimensional subcube also uniquely determines a representative hypercube.

By using a divide-and-conquer technique, our algorithm is based on the following two steps:

Step 1. Construct an embedding $\mathcal{F}_1$ that embeds $T_K(1)$ into the smallest possible hypercube such that $\mathcal{F}_1(r)$ has at least one idle neighbor. We consider an embedding $\mathcal{F}_1$ which embeds $T_K(1)$ into a hypercube as the inductive base.

For the propose of induction, we need an idle neighbor of the image of the tree root. Thus $\mathcal{F}_1$ will embed $T_K(1)$ into a $\lceil\log(K + 2)\rceil$ dimensional hypercube H''; i.e., $d(1) = \lceil\log(K + 2)\rceil$. *Let* v be an arbitrary vertex in H'' and u be adjacent vertex of v. We map the root of $T_K(1)$, r, into v and select arbitrary K vertices from the set Cs(v, $\lceil\log(K + 2)\rceil$, $\phi(K + 2)$) − $\{u, v\}$ as the images of the K leaves of $T_K(1)$. Thus, we have obtained a embedding f_1. It is not difficult to see that

(i) the dilation of $\mathcal{F}_1$ is bounded by $\phi(K + 2)$;
(ii) vertex u adjacent to $\mathcal{F}_1(r)$ $(= v)$ is idle.

Step 2. Assume that there exists an embedding $\mathcal{F}_h$ in which at least one of the neighbors of $\mathcal{F}_h(r)$ is idle, where $\mathcal{F}_h$ embeds $T_K(h)$ into a $d(h)$-dimensional hypercube with dilation dilation($\mathcal{F}_h$). Then we can construct a new embedding $\mathcal{F}_{h+1}$ which embeds $T_K(h + 1)$ into a $d(h) + \lceil\log K\rceil$-dimensional hypercube H' with at least one neighbor of $\mathcal{F}_{h+1}(r)$ idle. Let H^* be one of the representative hypercubes for a given K'-partition and let v_1 be a vertex of H^*. By Lemma 2.1, there exist $K - 1$ vertices, $v_2, \ldots, v_K$, in H^*, such that Max{dist(v_1, v_i) : $2 \leq i \leq K$} $\leq g(\lceil\log K\rceil, K) = \phi(K)$. Moreover, we assume that dist(v_1, v_2) = 1.

Let $H^{(1)}, H^{(2)}, \ldots, H^{(K)}$ be K $d(h)$-dimensional hypercubes corresponding to these K vertices $v_1, v_2, \ldots, v_K$. By the above inductive assumption, we can construct K embeddings $f_1, f_2, \ldots, f_K$ by duplicating $\mathcal{F}_h$ such that they satisfy the following four conditions:

(i) f_i embeds T_i into $H^{(i)}$;
(ii) $f_i(r_i) = v_i$;
(iii) u_i is an idle vertex adjacent to v_i under f_i;
(iv) $u_1, u_2, \ldots, u_K$ belong to the same representative hypercube, and

$$\text{dist}(u_1, u_i) = \text{dist}(v_1, v_i), \; 2 \leq i \leq K. \quad (2.3)$$

This configuration is illustrated in Fig. 2.1. Let s_i be an automorphic mapping from $H^{(i)}$ to $H^{(i)}$ such that $s_i(v_i) = u_i$. Based on these results, an embedding f_{h+1} that embeds $T_K(h + 1)$ into H' can be constructed as follows:

Step 2.1. The root r of $T_K(h + 1)$ is mapped to u_1, $\mathcal{F}_{h+1}(r) = u_1$;

Step 2.2. For $i > 2$, change the embedding f_i by applying s_i, *i.e.*, $\mathcal{F}_{h+1}(v) = s_i \circ f_i(v)$ for $v \in V(H^{(i)})$.

The construction of $\mathcal{F}_{h+1}$ is illustrated in Fig. 2.2.

It is not difficult to see that embedding $\mathcal{F}_{h+1}$ satisfies the following properties:

(i) $\mathcal{F}_{h+1}(r)$ $(= u_1)$ has at least one adjacent vertex idle, for example, u_2.

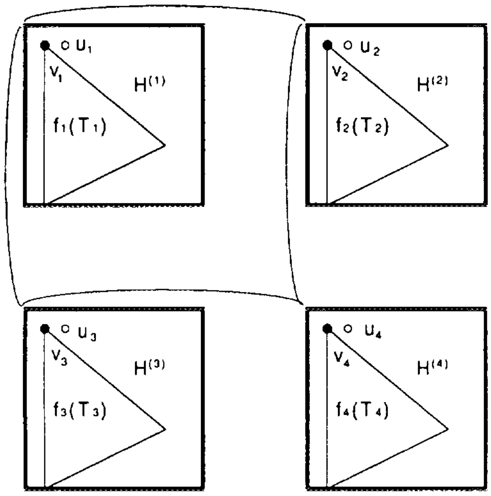

FIGURE 2.1

(ii) The maximum distance from $\mathcal{F}_{h+1}(r)$ to its child $\mathcal{F}_{h+1}(r_i)$ is no larger than $\text{Max}\{2, \phi(K)\}$.

Specifically, $\text{dist}(\mathcal{F}_{h+1}(r), \mathcal{F}_{h+1}(r_1)) = 1$, $\text{dist}(\mathcal{F}_{h+1}(r), \mathcal{F}_{h+1}(r_2)) = 2$, and $\text{dist}(\mathcal{F}_{h+1}(r), \mathcal{F}_{h+1}(r_i)) = \text{dist}(u_1, u_i) = \text{dist}(\nu_1, \nu_i)$, $3 \le i \le K$. By the selection of $\nu_1, \nu_2, \ldots, \nu_K$,

$$Max\{\text{dist}(\nu_1, \nu_i) : 3 \le i \le K\} \le \phi(K).$$

Since $\mathcal{F}_{h+1}$ does not change the topological structure of f_i within $H^{(i)}$, the dilation of $\mathcal{F}_{h+1}$ can be determined by

$$\text{dilation}(\mathcal{F}_{h+1}) = \text{Max}\{2, \phi(K), \text{dilation}(\mathcal{F}_h)\}. \quad (2.4).$$

Theorem 2.1. *A $T_K(h)$ can be embedded into a $d(h)$ dimensional hypercube with dilation* $\text{Max}\{2, \phi(K), \phi(K+2)\}$, *where* $d(h) = (h-1)\lceil \log K \rceil + \lceil \log(K+2) \rceil$ *and* $K \ge 3$.

Proof. It follows the above discussion. ∎

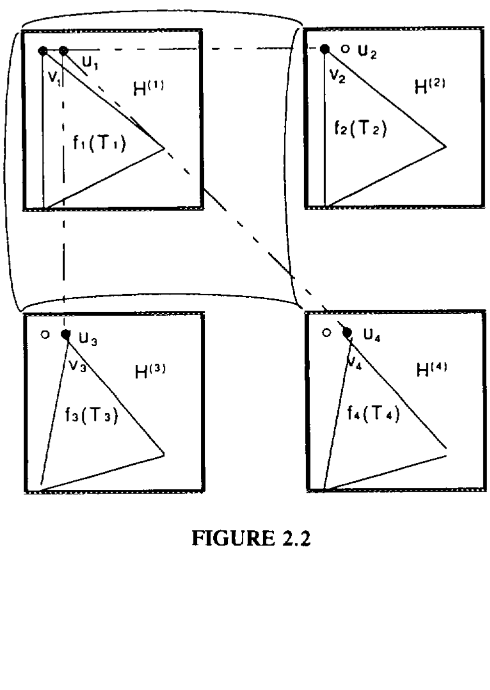

FIGURE 2.2

III. PRECISE EMBEDDING

3.1. A Lower Bound on Expansions

Precise embedding insists that dilation be equal to one by increasing expansions. The problem of precise embedding of a $T_K(h)$ into a hypercube H_d is equivalent to the problem of finding a subgraph of the hypercube which is isomorphic to $T_K(h)$. Here the question is: what is the smallest d of H_d such that the $T_K(h)$ can be precisely embedded. We assume that all embeddings discussed in this section are "precise embeddings."

Let H_d be such a hypercube with the smallest d. Note that for an arbitrary embedding f, if $f(r) = \nu$, then $f(x) \in \text{Cs}(\nu, d, h)$ for any $x \in V(T_K(h))$, because $\text{dist}(r, x) \le h$. From the previous discussion, we have

$$|Cs(\nu, d, h)| = \sum_{i=0}^{h} C_d^i \ge |V(T_K(h))| = (K^{h+1} - 1)/(K - 1). \quad (3.1)$$

Lemma 3.1. $C_d^h \le d^h/h!$.

Proof. By induction on d. The lemma is true when $d = h$. Since $C_{d+1}^h = (d+1)C_d^h/(d+1-h)$, by the inductive assumption, $C_{d+1}^h = (d+1)C_d^h/(d+1-h) \le (d+1)d^h/(d+1-h)h! \le (d+1)^h/h!$. ∎

Let H_d be the smallest hypercube into which a $T_K(h)$ $(K \ge 4)$ can be embedded. Since $K \ge 4$, by counting the number of vertices, it is not difficult to prove that $h < d/2$, which implies that $C_d^i \le C_d^h$, for any $i \le h$. Thus,

$$\sum_{i=0}^{h} C_d^i \le hC_d^h \le hd^h/h! = d^h/(h-1)!. \quad (3.2)$$

By (3.1), we have $d^h/(h-1)! \ge (K^{h+1} - 1)/(K-1) > K^h$. Thus,

$$d > K((h-1)!)^{1/h}. \quad (3.3)$$

From (3.3), we obtain an asymptotic estimation of this lower bound. By *Stirling Estimation*,

$$h! = \sqrt{2\pi h}\left(\frac{h}{e}\right)^h\left(1 + O\left(\frac{1}{h}\right)\right).$$

It is not difficult to see that $K[(h-1)!]^{1/h} \approx Kh/e$. In the following we are going to give a better estimate of the dimension d.

Theorem 3.1. *If a $T_K(h)$ can be embedded into a hypercube H_d, then $d \ge D(K, h)$, where* $D(K, h) = \lceil K[(1 - K^{-h-1})(h-1)!/(1 - K^{-2})]^{1/h} + 1 \rceil$.

Proof. Given a precise embedding f from a $T_K(h)$ to an H_d, let $f(r) = u$ and $u \in V(H_d)$, where r is the root of the $T_K(h)$. We construct the following subsets:

$$V_e(h) = \{v: v \in Cs(u, d, h) \text{ and } \mathrm{dist}(u, v) \text{ is even}\};$$

$$V_o(h) = \{v: v \in Cs(u, d, h) \text{ and } \mathrm{dist}(u, v) \text{ is odd}\};$$

$$U_e = \{v: v \in V(T_K(h)) \text{ and } \mathrm{dist}(r, v) \text{ is even}\};$$

$$U_o = \{v: v \in V(T_K(h)) \text{ and } \mathrm{dist}(r, v) \text{ is odd}\}.$$

Since both $T_K(h)$ and H_d are bipartite graphs, and $f(T_K(h))$ is a subgraph of H_d, we have $f(U_e) \subseteq V_e(h)$ and $f(U_o) \subseteq V_o(h)$, which implies that

$$|U_e| = |f(U_e)| \le |V_e(h)| \quad \text{and} \quad |U_o| = |f(U_o)| \le |V_o(h)|. \tag{3.4}$$

When h is even, $h = 2m$; then

$$|V_e(h)| = \sum_{i=0}^{m} C_d^{2i} = \sum_{i=0}^{h} C_{d-1}^{i},$$

$$|V_o(h)| = \sum_{i=1}^{m} C_d^{2i-1} = \sum_{i=0}^{h-1} C_{d-1}^{i};$$

$$|U_e| = (K^{h+2} - 1)|(K^2 - 1),$$

$$|U_o| = K(K^h - 1)/(K^2 - 1).$$

Taking $|f(U_e)| \le |V_e(h)|$ as a requirement, we have

$$\sum_{i=0}^{h} C_{d-1}^{i} \ge (K^{h+2} - 1)/(K^2 - 1). \tag{3.5}$$

When h is odd, $h = 2m + 1$; then

$$|V_e(h)| = \sum_{i=0}^{m} C_d^{2i} = \sum_{i=0}^{h-1} C_{d-1}^{i},$$

$$|V_o(h)| = \sum_{i=1}^{m} C_d^{2i+1} = \sum_{i=0}^{h} C_{d-1}^{i};$$

$$|U_e| = (K^{h+1} - 1)/(K^2 - 1),$$

$$|U_o| = K(K^{h+1} - 1)/(K^2 - 1).$$

Taking $|f(U_o)| \le |V_o(h)|$ as a requirement, we have

$$\sum_{i=0}^{h} C_{d-1}^{i} \ge K(K^{h+1} - 1)/(K^2 - 1). \tag{3.6}$$

From the inequalities (3.2), (3.5), and (3.6), we have

$$d \ge \begin{cases} K[(1 - K^{-h-2})(h-1)!/(1 - K^{-2})]^{1/h} + 1, & h \text{ is even} \\ K[(1 - K^{-h-1})(h-1)!/(1 - K^{-2})]^{1/h} + 1, & h \text{ is odd.} \end{cases}$$

Therefore, $d \ge D(K, h) = \lceil K[(1 - K^{-h-1})(h - 1)!/(1 - K^{-2})]^{1/h} + 1 \rceil$. ■

Thus, a lower bound on the expansion of precise embeddings is $(K - 1)2^{D(K,h)}/(K^{h+1} - 1)$. Note that $D(K, h)$ is a better bound than that in (3.3).

3.2. A Precise Embedding with a Small Expansion

It is trivial to embed a $T_K(h)$ into a $(Kh + 1)$-dimensional hypercube. We will present a nontrivial embedding which embeds a $T_K(h)$ into a $(K - 1)h + 1$ dimensional hypercube ($K \ge 3$). It seems difficult to determine, and remains to be seen, whether we can find a precise embedding such that the dimension of the hypercube is smaller than Kh by a multiplicative factor. (Note that Kh/e is a lower bound.)

Our embedding method is based on an induction on h. Let $H_{c(h)}$ be the smallest hypercube a $T_K(h)$ can be embedded into. We will show how to precisely embed a $T_K(h + 1)$ into a $(K - 1) + c(h)$-dimensional hypercube. Let $H^{\sim}$ be a $(K - 1) + c(h)$ dimensional hypercube. It can be partitioned into 2^{K-1} $c(h)$-dimensional subcubes, $\{H[0], H[1], \ldots, H[2^{K-1} - 1]\}$, where $H[i]$ is a subcube in which the first $K - 1$ bits of any vertex label form the binary number i, $0 \le i \le 2^{K-1} - 1$. Two such subcubes $H[i]$ and $H[j]$ are called adjacent if i and j differ in exactly one bit. Given one of these $c(h)$-dimensional subcubes, say $H^{(1)}$, there are a total of $(K - 1)$ different adjacent $c(h)$-dimensional subcubes, denoted as $H^{(2)}, \ldots, H^{(K)}$. Let $H^{\#}$ be another $c(h)$-dimensional subcube which is adjacent to $H^{(2)}$ but not adjacent to $H^{(1)}$. The outline of the proposed algorithm is as follows:

Step 1. Construct an embedding $\mathcal{I}_1$ to embed $T_K(1)$ into the smallest hypercube H_K. The embedding $T_K(1)$ into a K-dimensional hypercube is trivial. The root of $T_K(1)$ r has to be connected with its K children, which implies that the image of r, $\mathcal{I}_1(r)$, must have K neighbors. Therefore, it is impossible to embed $T_K(1)$ into a hypercube whose dimension is less than K. We have $c(1) = K$.

Step 2. Assume that there exists an embedding $\mathcal{I}_h$ that embeds $T_K(h)$ into a $c(h)$-dimensional hypercube for $h \ge 1$. Now we construct a new embedding $\mathcal{I}_{h+1}$ of a $T_K(h + 1)$ into a $(K - 1 + c(h))$-dimensional hypercube $H^{\sim}$. Let the $T_K(h + 1)$ be labeled as in Section 2.2, where r is the root, and r_2 is a child of root r. Let r' be one of children of r_2 which is also a root of a subtree $T_K(h - 1)$, T' (see Fig. 3.1). By the inductive assumption, we can construct K embeddings $g_1, g_2, \ldots, g_K$ by duplicating $\mathcal{I}_h$ such that

(i) g_i embeds T_i into $H^{(i)}$, $1 \le i \le K$,
(ii) $g_i(r_i) = v_i$, $v_i \in H^{(i)}$,
(iii) $(v_1, v_i) \in E(H^{\sim})$, $2 \le i \le K$,
(iv) $g_2(r') = u_2$, and u_2 is adjacent to vertex u_1 in $H^{(1)}$, $(v_1, u_1) \in E(H^{(1)})$.

These mappings are illustrated in Fig. 3.1.

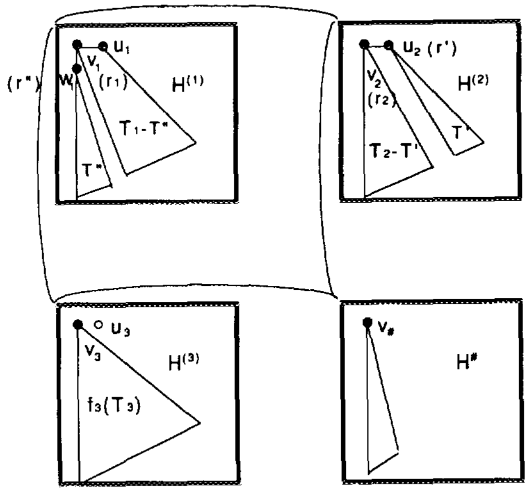

FIGURE 3.1

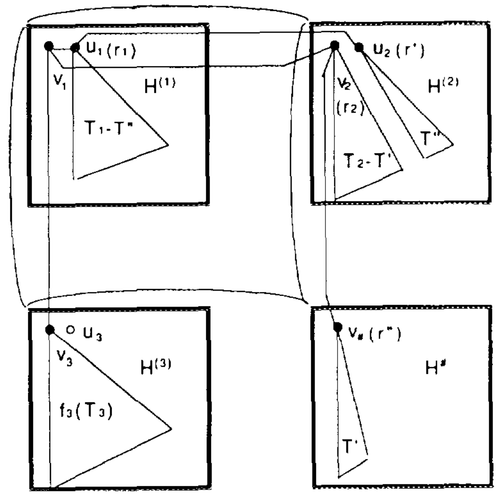

FIGURE 3.2

Now we distinguish two cases.

Case 1. ν_1 has an idle neighbor w_1 under g_1.

For this case, we introduce an automorphic mapping s_1 from $V(H^{(1)})$ to itself such that $s_1(\nu_1) = w_1$ and $s_1(w_1) = \nu_1$. Then $g_1' = s_1 \circ g_1$ embedding T_1 into $H^{(1)}$ so that ν_1 is idle and $g_1'(r_1) = w_1$. Now we map the root r of the $T_K(h + 1)$ to ν_1, which is denoted $g_0(r) = \nu_1$. The $\mathcal{I}_{h+1}$ is obtained by the union of $g_0, g'_1, g_2, \ldots, g_K$.

Case 2. ν_1 does not have any idle neighbor under g_1.

In order to make room for the root r of $T_K(h + 1)$, we will move the image of one subtree of T_2 to $H^{\#}$, and move the image of one subtree of T_1 to the vacant place in $H^{(2)}$. Specifically, we follow the following steps (see Fig. 3.2):

Step 2.1. Choose one subtree T'' of T_1 rooted at r'' and one subtree T' of T_2 rooted at r' (see Fig. 3.1) such that they are embedded by g_1 and g_2 and $g_1(r'')$ and $g_2(r')$ are adjacent in H~.

Step 2.2. Move the image of T' from $H^{(2)}$ to $H^{\#}$ such that r' is mapped to $\nu^{\#}$ which is adjacent to ν_2. In other words, we map T' into $H^{\#}$, instead of $H^{(2)}$.

Step 2.3. Move the image of T'' from $H^{(1)}$ to the vacant place in $H^{(2)}$ by Step 2.2.

Step 2.4. Now ν_1 has an idle neighbor r''. Make the same modification on g_1 as in Case 1, and map r to ν_1.

It is easy to see that $\mathcal{I}_{h+1}$ is a precise embedding of $T_K(h + 1)$ into an $H^{\sim}$, since $\text{dist}(\mathcal{I}_{h+1}(r), \mathcal{I}_{h+1}(r_i)) = 1$ for $i = 1, 2, \ldots, K$, $\text{dist}(\mathcal{I}_{h+1}(r_1), \mathcal{I}_{h+1}(r'')) = 1$, and $\text{dist}(\mathcal{I}_{h+1}(r_2), \mathcal{I}_{h+1}(r')) = 1$. The other topological relations between vertices have been maintained.

THEOREM 3.2. *A $T_K(h)$ can be precisely embedded into a $(K - 1)h + 1$-dimensional hypercube.*

Proof. The above algorithm provides an inductive proof for this theorem. We only need to point out that $c(h)$ can be obtained by the following recursive relations:

$$\begin{cases} c(h) = c(h - 1) + (K - 1), & h > 1 \\ c(1) = K, & \text{otherwise.} \end{cases}$$

Thus, $c(h) = (K - 1)h + 1$. ■

So $(K - 1)h + 1$ is an upper bound on the dimension d of a H_d such that $T_K(h)$ can be precisely embedded. There

TABLE I

	Complete binary tree	Complete quadtree	K-complete tree
Dilated Embedding			
Hypercube's dimension			
Lower bound	$h + 1$	$2h + 1$	$\lceil \log Kh + 1 \rceil$
Upper bound	$h + 1$	$2h + 1$	$\lceil \log K \rceil (h - 1) + \lceil \log(K + 2) \rceil$
Dilation			
Lower bound	2	2	$\lceil \phi(K + 1) \rceil$
Upper bound	2	2	$\max \{2, \phi(K), \phi(K + 2)\}$
Precise Embedding			
Hypercube's dimension			
Lower bound	$h + 2$	N/A	$\max \{\lceil h \log K \rceil, D(K, h)\}$
Upper bound	$h + 2$	N/A	$(K - 1)h + 1$

still exists plenty of room between this upper bound $(K-1)h+1$ and the lower bound $D(K, h)$. The following table lists some values of these bounds:

K	h	$\lceil K[(h-1)!]^{1/h}\rceil$	$\lceil Kh/e\rceil$	$D(K, h)$	$(K-1)h+1$
4	3	6	5	7	10
5	9	17	17	20	37
7	15	38	39	40	91
20	15	108	111	109	286

It must be mentioned that these lower bounds do not include the case $K \leq 3$. For the binary complete trees, a tight lower bound is $h+2$. By (2.1), a lower bound for 3-ary complete trees is $\lceil h \log 3\rceil$ which is a better bound than $D(3, h)$ for 3-ary complete trees.

IV. CONCLUSION

Both dilated embedding and precise embedding of a $T_K(h)$ into a hypercube are explored in this paper. Our results and some previous results, which are listed in Table 1, are nearly optimal. However, there is still a room for further improvement.

REFERENCES

1. Hong, J.-W., Mehlhorn, K., and Rosenberg, A. L. Cost tradeoffs in graph embeddings. *J. Assoc. Comput. Mach.* **30**, 4 (1983), 709–728.
2. Leighton, F. T. *Introduction to Parallel Algorithms and Architectures: Arrays, Trees and Hypercubes*, Morgan Kaufmann, San Mateo, CA, 1992.
3. Saad, Y., and Schultz, M. H. Topological properties of hypercubes. *IEEE Trans. Computers.* **C**-37, 7 (1988), 867–872.
4. Havel, I., and Liebl, P. Embedding the polytomic tree into the n-cube. *Casopis Pěst. Mat.* **98** (1973), 307–314.
5. Wu, A. Y. Embedding of tree networks into hypercubes. *J. Parallel Distrib. Comput.* **2** (1985), 238–249.
6. Leiss, E. L., and Reddy, H. N. Embedding complete binary trees into hypercubes. *Inform. Process. Lett.* **38** (1991), 197–199.
7. Ho, C. T., and Johnson, S. L. Dilation and embedding of a hyperpyramid into a hypercube. In *Proceedings of the Supercomputing '89*. Nov. 1989, pp. 294–303.
8. Krishnakumar, N., Hedge, V., and Iyengar, S. S. Fault tolerance based embeddings of quadtrees into hypercubes. In *Proceedings of the 1991 International Conference on Parallel Processing*. Vol. III, pp. 244–249.
9. Bhatt, S., Chung, F., Leighton, T, and Rosenberg, A. Optimal simulations of tree machines. In *Proceedings of the 27th Annual Symposium on Foundations of Computer Science*. IEEE, New York, Oct. 1986, pp. 274–282.
10. Monien, B., and Sudborough, H. Simulating binary trees on hypercubes. In *Lecture Notes in Computer Science*. Springer-Verlag, Berlin/New York, 1988, Vol. 319, pp. 170–180.
11. Wagner, A. Embedding trees in the hypercube. *Report* 204/87, Univ. of Toronto, 1987.
12. Wagner, A., and Corneil, D. Embedding trees in a hypercube is NP-complete. *SIAM J. Comput.* **19**, 3 (June 1990), 570–590.
13. Akers, S. B., Harel, D., and Krishnamurthy, B. The star graph: An attractive alternative to n-cube. In *International Conference on Parallel Processing*. 1987, pp. 393–400.

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