Optimal Path Embedding in Crossed Cubes

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Abstract—The crossed cube is an important variant of the hypercube. The n-dimensional crossed cube has only about half diameter, wide diameter, and fault diameter of those of the n-dimensional hypercube. Embeddings of trees, cycles, shortest paths, and Hamiltonian paths in crossed cubes have been studied in literature. Little work has been done on the embedding of paths except shortest paths, and Hamiltonian paths in crossed cubes. In this paper, we study optimal embedding of paths of different lengths between any two nodes in crossed cubes. We prove that paths of all lengths between $\lceil \frac{n+1}{2} \rceil + 1$ and $2^n - 1$ can be embedded between any two distinct nodes with a dilation of 1 in the n-dimensional crossed cube. The embedding of paths is optimal in the sense that the dilation of the embedding is 1. We also prove that $\lceil \frac{n+1}{2} \rceil + 1$ is the shortest possible length that can be embedded between arbitrary two distinct nodes with dilation 1 in the n-dimensional crossed cube.

Index Terms—Crossed cube, graph embedding, optimal embedding, interconnection network, parallel computing system.

1 Introduction

Interconnection networks play an important role in parallel computing systems. An interconnection network can be represented by a graph $G = (V, E)$, where V represents the node set and E represents the edge set. In this paper, we use graphs and interconnection networks (networks for short) interchangeably.

Graph embedding is a technique in parallel computing that maps a guest graph into a host graph (usually an interconnection network). There are many applications of graph embedding, such as architecture simulation, processor allocation, VLSI chip design, etc. Architecture simulation is the simulation of one architecture by another. This can be modeled as a graph embedding, which embeds the guest architecture (represented as a graph) into the host architecture, where the nodes of the graph represent the processors and the edges of the graph represent the communication links between the processors [22], [25], [33], [34]. In parallel computing, a large process is often decomposed into a set of small subprocesses that can execute in parallel with communications among these subprocesses. According to these communication relations among these subprocesses, a graph can be obtained, in which the nodes in the graph represent the subprocesses and the edges of the graph represent the communication links between these subprocesses [8]. Thus, the problem of allocating these subprocesses into a parallel computing systems can be again modeled as a graph embedding problem. The problem of laying out circuits on VLSI chips can also be reduced to graph embedding problems [3], [30], [31].

Graph embedding can be formally defined as: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an embedding from G_1 to G_2 is an injective mapping $\psi : V_1 \rightarrow V_2$. We call G_1 the *guest* graph and G_2 the *host* graph. An important performance metric of embedding is dilation. The dilation of embedding ψ is defined as

$$\mathrm{dil}(G_1, G_2, \psi) = \max\{\mathrm{dist}(G_2, \psi(u), \psi(v)) | (u, v) \in E_1\},$$

where $\mathrm{dist}(G_2, \psi(u), \psi(v))$ denotes the distance between the two nodes $\psi(u)$ and $\psi(v)$ in G_2. The smaller the dilation of an embedding is, the shorter the communication delay that the graph G_2 simulates the graph G_1.

Embedding ψ from G_1 to G_2 is *optimal* if ψ has the smallest dilation among all the embeddings from G_1 to G_2. Clearly, $\mathrm{dil}(G_1, G_2, \psi) \geq 1$. When $\mathrm{dil}(G_1, G_2, \psi) = 1$, ψ is surely optimal and G_1 is a subgraph of G_2 in this case. Finding the optimal embedding of graphs is NP-hard.

Most of the works on graph embedding consider paths, trees, meshes, and cycles as guest graphs because they are the architectures widely used in parallel computing systems [1], [2], [6], [16], [19], [20], [21], [22], [23], [25], [32], [33], [34]. In this paper, we will study the path embedding problem. Some special paths take important roles in parallel computing. A shortest path (routing) between two nodes in an interconnection network is an optimal communication path in terms of delay. Edge-disjoint paths between two nodes are fundamental of routing in high-speed networks [5], [26], while node-disjoint paths are significant for fault-tolerant routing [17], [18]. A longest path—Hamiltonian path can be used in dual-path and multipath multicast routing algorithms to alleviate congestion or avoid deadlock incurred by traditionally tree-based multicast algorithms in parallel computing systems [4], [27], [28]. This paper addresses the issue of path embedding in crossed cubes, which takes paths as guest graphs and crossed cubes as host graphs.

The crossed cube is an important variant of the hypercube [9]. It has drawn a great deal of attention [7], [9], [10], [11], [12], [14], [15], [23], [24], [25], [33]. It has the same node number and edge number as the hypercube with the same dimension, but has only about half the diameter [7], [9], wide diameter [7], and fault diameter [7] of those of the hypercube. Embedding of shortest paths, Hamiltonian paths, cycles, and trees as guest graphs into crossed cubes

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were studied in literature [7], [9], [23], [25], [33]. In [7], [9], Efe and Chang et al., respectively, provided an embedding of shortest paths (i.e., shortest routing algorithms) between any two nodes in the n-dimensional crossed cube. They also showed that cycles of all lengths from 4 to 2^n can be embedded into the n-dimensional crossed cube. Huang et al. gave fault-tolerant Hamiltonian path embedding in [23]. Yang et al. further provided fault-tolerant cycle embedding in the n-dimensional crossed cube [33]. In the case of tree embedding, Kulasinghe and Bettayeb gave a perfect result. They proved that a complete binary tree of $2^n - 1$ nodes can be embedded into the n-dimensional crossed cube [25]. All these embeddings in crossed cubes have dilations 1 and, thus, they are all optimal.

There is little work reported on path embedding in crossed cubes in literature. The most notable work on path embedding in crossed cubes is the shortest path embedding (shortest routing) [7], [9] and the Hamiltonian path embedding [23]. In this paper, we will discuss the optimal embedding of paths of different lengths between any two nodes in crossed cubes. The original contributions of this paper are as follows:

1. *For any two distinct nodes x and y in the n-dimensional crossed cube, and any integer l with $\lceil \frac{n+1}{2} \rceil + 1 \leq l \leq 2^n - 1$, a path of length l can be embedded between x and y with a dilation of 1 in the n-dimensional crossed cube. The embedding is optimal in the sense that its dilation is 1.*
2. *There exist two distinct nodes x and y in the n-dimensional crossed cube such that no path of length $\lceil \frac{n+1}{2} \rceil$ can be embedded between x and y with a dilation of 1 in the n-dimensional crossed cube. This conclusion demonstrates that $\lceil \frac{n+1}{2} \rceil + 1$ is the shortest possible length that can be embedded between arbitrary two distinct nodes with a dilation of 1 in the n-dimensional crossed cube.*

In interconnection networks, the Hamilton-connectivity (i.e., any two distinct nodes are connected by a Hamiltonian path) is an important property. The results obtained in this paper show a stronger connectivity for crossed cubes. We prove that, in the n-dimensional crossed cube, any two distinct nodes are connected not only by a Hamiltonian path, but also by many other paths of consecutive lengths. This property is not well-known in the existing interconnection networks.

The rest of this paper is organized as follows: Section 2 provides the preliminaries. Section 3 discusses embedding paths of lengths from $\lceil \frac{n+1}{2} \rceil + 1$ through $2^n - 1$ in the n-dimensional crossed cube. Section 4 proves that the existence of two nodes that no path of length $\lceil \frac{n+1}{2} \rceil$ can be embedded between these two nodes with a dilation of 1 in the n-dimensional crossed cube. In Section 5, we give conclusions.

2 PRELIMINARIES

Let $V(G)$ and $E(G)$ denote the node set and the edge set of a graph G, respectively. Given $V' \subseteq V(G)$, the subgraph induced by V' in G is denoted by $G[V']$. A path P between node u and node v in G is denoted by $P: u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$. Nodes u and v are the two end nodes of path P. The length of path P is denoted by $\text{len}(P)$. Path P can also be denoted by: $u = u^{(0)}, u^{(1)}, \ldots, u^{(i-1)}, P_1, u^{(j+1)}, u^{(j+2)}, \ldots, u^{(k)} = v$, where the path P_1 is called a *subpath* of P between $u^{(i)}$ and $u^{(j)}$, i.e., $u^{(i)}, u^{(i+1)}, \ldots, u^{(j)}$ $(i \leq j)$. The path P_1, starting from $u^{(i)}$ and ending with $u^{(j)}$, can be denoted by $\text{path}(P, u^{(i)}, u^{(j)})$. For the path P, if $u = v$ $(k \geq 3)$, then P is a cycle. If (x, y) is an edge in a cycle C, we use $C - (x, y)$ to denote the path after deleting the edge (u, v) in C.

Let $\text{dist}(G, x, y)$ denote the distance between two nodes x and y of G. The diameter of G is defined as $\text{diam}(G) = \max\{\text{dist}(G, x, y) | x, y \in V(G), x \neq y\}$.

If x is a node in the path P, then we denote it as $x \in P$. Otherwise, it is denoted as $x \notin P$. Similarly, if (x, y) is an edge in the path P, then we denote it as $(x, y) \in P$. Otherwise, it is denoted as $(x, y) \notin P$. If $V' \subseteq V(G)$ and no node in V' is in P, then we write as $V' \bigcap P = \emptyset$.

A binary string x of length n is denoted by $x_{n-1}x_{n-2} \ldots x_1x_0$, where x_{n-1} is the most significant bit and x_0 is the least significant bit. The ith bit x_i of x can also be written as $\text{bit}(x, i)$. The complement of x_i is denoted by $\overline{x_i}$. Letting z be a binary string of length k, the binary string of length ik obtained by concatenating one by one i strings z is denoted by z^i.

The n-dimensional crossed cube (denoted by CQ_n) is an n-regular graph that contains 2^n nodes and $n2^{n-1}$ edges. Every node of CQ_n is identified by a unique binary string, which is also called *address*, of length n. In this paper, we would not distinguish between nodes and their binary addresses. The n-dimensional crossed cube can be recursively defined as follows [9], [25].

Definition 1. *Two binary strings, $x = x_1x_0$ and $y = y_1y_0$, of length two are said to be pair related (denoted by $x \sim y$) if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.*

Definition 2. *CQ_1 is the complete graph on two nodes whose addresses are 0 and 1. CQ_n consists of two subcubes CQ_{n-1}^0 and CQ_{n-1}^1. The most significant bit of the addresses of the nodes of CQ_{n-1}^0 and CQ_{n-1}^1 are 0 and 1, respectively. The nodes $u = u_{n-1}u_{n-2} \ldots u_1u_0$ and $v = v_{n-1}v_{n-2} \ldots v_1v_0$, where $u_{n-1} = 0$ and $v_{n-1} = 1$, are joined by an edge in CQ_n if and only if*

1. *$u_{n-2} = v_{n-2}$ if n is even, and*
2. *$u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$, for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.*

Given $x, y \in V(CQ_n)$, $0 \leq i \leq n-1$ and $n \geq 1$, if $x_j = y_j$ for all $j \in \{i+1, i+2, \ldots, n-1\}$, $\text{bit}(x, i) = \overline{\text{bit}(y, i)}$ and $(x, y) \in E(CQ_n)$, then we write as $x \overset{i}{\sim} y$. By Definition 2, if $(x, y) \in E(CQ_n)$, then there exists an i such that $0 \leq i \leq n-1$ and $x \overset{i}{\sim} y$, and for any j with $0 \leq j \leq n-1$, there exists $w \in V(CQ_n)$ such that $x \overset{j}{\sim} w$. If $k \geq 1$, $n \geq 2$, and y is a binary string of length k, let $V' = \{yx | x \text{ is a binary string of length } n-k\}$. The subgraph induced by V' in CQ_n, i.e., $CQ_n[V']$, is written as CQ_{n-k}^y. Obviously, by Definition 2, CQ_{n-k}^y is isomorphic to CQ_{n-k}.

In Fig. 1, Fig. 1a, and Fig. 1b are two different drawings of CQ_3; Fig. 1c is CQ_4. We can easily find the symmetric property of CQ_3.

In [9], Efe gave an $O(n^2)$ algorithm to find a shortest path between any two distinct nodes in CQ_n. In [7], Chang et al. improved this algorithm. They gave an $O(n)$ algorithm, which we call *CSH algorithm*, to get more shortest paths between any two distinct nodes in CQ_n. CSH algorithm introduces definitions of *distance-preserving pair related* and

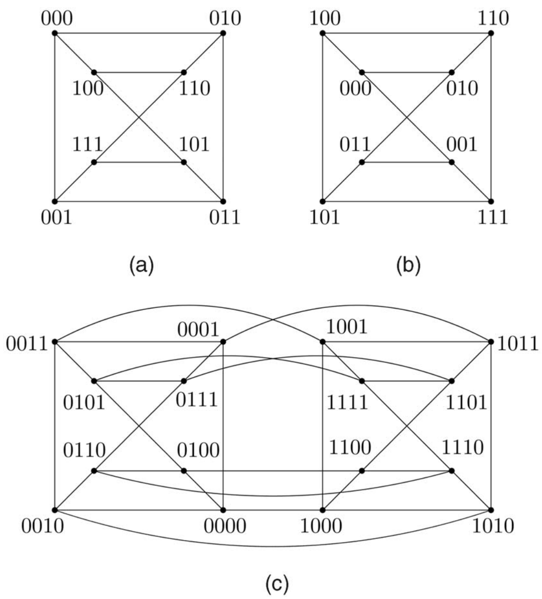

Fig. 1. (a) and (b) Two different drawings of CQ_3. (c) CQ_4.

pair related distance. These two definitions are iterated as follows [7]:

Let u and v be two distinct nodes in CQ_n. The ith *double bit* of nodes u is defined as a 2-bit string $u_{2i+1}u_{2i}$ for $0 \le i \le \lfloor \frac{n}{2} \rfloor - 1$, and as simply a single bit u_{2i} for $i = \lfloor \frac{n}{2} \rfloor$ and n odd. Bit l is called the *most significant differing bit* between u and v if $u_s = v_s$ for all s with $l+1 \le s \le n-1$ and $u_l \ne v_l$. Let $i^* = \lfloor \frac{l}{2} \rfloor$ be called the *most significant differing double bit*. A function ρ on u, v is defined as follows:

$$\rho_j(u,v) = 0 \text{ for all } j \ge i^* + 1,$$
$$\rho_{i^*}(u,v) = \begin{cases} 2 & \text{if } u_{2i^*+1}u_{2i^*} = \overline{v}_{2i^*+1}\overline{v}_{2i^*}, \\ 1 & \text{otherwise.} \end{cases}$$

Further, for $j \le i^* - 1$, $\rho_j(u,v)$ can be defined using the notion of *distance-preserving pair related* (abbreviated as *d.p. pair related*) as follows:

Definition 3. *$u_{2j+1}u_{2j}$ and $v_{2j+1}v_{2j}$, for $j \le i^* - 1$, are distance-preserving pair related if one of the following conditions holds:*

1. $(u_{2j+1}u_{2j}, v_{2j+1}v_{2j}) \in \{(01,01),(11,11)\}$ *and* $\sum_{k=j+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_k(u,v)$ *is even,*
2. $(u_{2j+1}u_{2j}, v_{2j+1}v_{2j}) \in \{(01,11),(11,01)\}$ *and* $\sum_{k=j+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_k(u,v)$ *is odd, and*
3. $(u_{2j+1}u_{2j}, v_{2j+1}v_{2j}) \in \{(00,00),(10,10)\}$.

We write $u_{2j+1}u_{2j} \overset{d.p.}{\sim} v_{2j+1}v_{2j}$ if $u_{2j+1}u_{2j}$ and $v_{2j+1}v_{2j}$ are d.p. pair related, and $u_{2j+1}u_{2j} \overset{d.p.}{\nsim} v_{2j+1}v_{2j}$ otherwise.

Then, $\rho_j(u,v)$ for $j \le i^* - 1$ is recursively defined as follows:

$$\rho_j(u,v) = \begin{cases} 0 & \text{if } u_{2j+1}u_{2j} \overset{d.p.}{\sim} v_{2j+1}v_{2j}, \\ 1 & \text{otherwise.} \end{cases}$$

The pair related distance between u and v is defined, denoted by $\rho(u,v)$, as

$$\rho(u,v) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_j(u,v).$$

Furthermore, Chang et al. proved an important result as follows:

$$\text{dist}(CQ_n, u, v) = \rho(u,v).$$

This result will be used in the proofs of this paper.

For details on the CSH algorithm, see [7].

3 Embedding Paths of Lengths from $\lceil \frac{n+1}{2} \rceil + 1$ to $2^n - 1$

In this section, we will prove that paths of lengths from $\lceil \frac{n+1}{2} \rceil + 1$ through $2^n - 1$ can be embedded between any two distinct nodes with a dilation of 1 in CQ_n for $n \ge 3$. In other words, there exists paths of all lengths from $\lceil \frac{n+1}{2} \rceil + 1$ through $2^n - 1$ between any two distinct nodes in CQ_n for $n \ge 3$. The following lemma shows the diameter of CQ_n.

Lemma 1 [7], [9]. *If $n \ge 1$, then* $\text{diam}(CQ_n) = \lceil \frac{n+1}{2} \rceil$.

In fact, arbitrarily selecting two distinct nodes in CQ_n, we have $|V(CQ_n)| = 2^n$. Thus, in this problem, we have to consider $\binom{2^n}{2}(2^n - \lceil \frac{n+1}{2} \rceil - 1)$ paths of all lengths from $\lceil \frac{n+1}{2} \rceil + 1$ through $2^n - 1$ between x and y in CQ_n. Even given a little integer $n = 10$, one has to consider more than 10^8 paths between x and y. This is not realistic. In order to simplify the proof, we adopt the induction on the dimension n of the crossed cube to prove this result in Theorem 1. Before beginning this proof, we need to give some preliminary lemmas. The following lemma is on the existence of a special cycle of length 4.

Lemma 2. *If $n \ge 3$ and $x \overset{n-1}{\text{—}} y$ for $x, y \in V(CQ_n)$, letting $x \overset{1}{\text{—}} u$ and $y \overset{1}{\text{—}} v$, then $(u,v) \in E(CQ_n)$ and, thus, $C : x, u, v, y, x$ is a cycle of length 4 that contains the edge (x,y) in CQ_n.*

Proof. Let $x = 0x_{n-2}\ldots x_1x_0$, $y = 1y_{n-2}\ldots y_1y_0$. Then, by the definition of the crossed cube, we have $(x_1x_0, y_1y_0) \in \{(00,00),(10,10),(01,11),(11,01)\}$. In fact, we need only verify the truth of the lemma for $(x_1x_0, y_1y_0) \in \{(00,00),(01,11)\}$. That is easy and, therefore, omitted. □

To find the embedding of paths of lengths from $\lceil \frac{n+1}{2} \rceil + 1$ to $2^n - 1$, three special cases need to be dealt with separately. The three special cases are when n is odd and: 1) $\text{dist}(CQ_n, x, y) = \lceil \frac{n+1}{2} \rceil$ and $n \ge 5$; 2) $\text{dist}(CQ_n, x, y) = \lceil \frac{n+1}{2} \rceil - 1$ and $n \ge 5$; 3) $\text{dist}(CQ_n, x, y) = \lceil \frac{n+1}{2} \rceil - 2$ and $n \ge 7$. The results for the three cases are given, respectively, in Lemmas 3, 4, and 7. They will be used in the proof of Theorem 1.

Lemma 3. *If $n \ge 5$ and n is odd, $x = 0x_{n-2}\ldots x_1x_0 \in V(CQ^0_{n-1})$ and $y = 1y_{n-2}\ldots y_1y_0 \in V(CQ^1_{n-1})$ with $\overline{x_{n-2}}x_{n-3} \overset{d.p.}{\sim} y_{n-2}y_{n-3}$ and $x_{n-2}x_{n-3} \overset{d.p.}{\sim} \overline{y_{n-2}}y_{n-3}$, and $\text{dist}(CQ_n, x, y) = \lceil \frac{n+1}{2} \rceil$, then there exists a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_n.*

Proof. Notice that $\lceil \frac{n+1}{2} \rceil - 1 = \lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. By the definition of function $\rho_i(u,v)$, we have $\rho_i(x,y) = 1$, $i = 0, 1, \ldots, \lceil \frac{n+1}{2} \rceil - 1$. For x_1x_0 and y_1y_0, we need only deal with the cases that $x_1x_0 = \overline{y_1y_0}$, $x_1x_0 = \overline{y_1}y_0$, $x_1x_0 = y_1\overline{y_0}$, and $x_1x_0 = y_1y_0$.

Case 1. $x_1x_0 = \overline{y_1y_0}$. We have the cases as below.

Case 1.1. $x_1x_0 \in \{01, 11\}$. Then, $y_1y_0 \in \{10, 00\}$. Let $x \overset{0}{\text{—}} u \overset{1}{\text{—}} v$. Then, $\text{bit}(v,1)\text{bit}(v,0) = y_1y_0 \in \{10,00\}$. Therefore, $\rho_0(v,y) = 0$. Obviously, $\rho_i(v,y) = \rho_i(x,y) = 1$ for

$i \in \{1, 2, \ldots, \lceil\frac{n+1}{2}\rceil - 1\}$ and, thus, $\text{dist}(CQ_n, v, y) = \lceil\frac{n+1}{2}\rceil - 1$. By using the CSH algorithm, we can get a shortest path P between v and y in CQ_n and $\text{dist}(CQ_n, x, y) = \text{dist}\ (CQ_n, u, y) \neq \text{dist}(CQ_n, z, y)$ for every node z in P. So, $\{x, u\} \bigcap P = \emptyset$. Then,

$$x, u, P$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 1.2. $x_1x_0 \in \{10, 00\}$. Then, $y_1y_0 \in \{01, 11\}$. Exchanging the position of x and y, this case is actually reduced to Case 1.1.

Case 2. $x_1x_0 = \overline{y_1}y_0$. We have the following cases.

Case 2.1. $x_1x_0 \in \{00, 10\}$. Then, $y_1y_0 \in \{10, 00\}$. Let $x \overset{0}{\text{—}} u \overset{n-2}{\text{—}} v \overset{0}{\text{—}} w$. Then, $\text{bit}(w,1)\text{bit}(w,0) = y_1y_0 \in \{00, 10\}$ and $\rho_0(w, y) = \rho_{\lceil\frac{n+1}{2}\rceil-2}(w, y) = 0$. Clearly, $\rho_i(w, y) = \rho_i(x, y) = 1$ for $i \in \{0, 1, \ldots, \lceil\frac{n+1}{2}\rceil - 1\} - \{0, \lceil\frac{n+1}{2}\rceil - 2\}$. As a result, $\text{dist}(CQ_n, w, y) = \lceil\frac{n+1}{2}\rceil - 2$. By using CSH algorithm, we can get a shortest path P, whose length is $\lceil\frac{n+1}{2}\rceil - 2$, between w and y in CQ_n. We can easily verify that for any $z \in P$ and $z' \in \{x, u, v\}$, $\text{dist}(CQ_n, z, y) \neq \text{dist}(CQ_n, z', y)$. Therefore, $\{x, u, v\} \bigcap P = \emptyset$. Then,

$$x, u, v, P$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 2.2. $x_1x_0 \in \{01, 11\}$. Then, $y_1y_0 \in \{11, 01\}$. Let $x \overset{n-2}{\text{—}} u \overset{0}{\text{—}} v$ and $w \overset{0}{\text{—}} y$. Then,

$$\begin{aligned}\text{bit}(v,1)\text{bit}(v,0) &= \text{bit}(w,1)\text{bit}(w,0)\\ &= \overline{x_1x_0} = y_1\overline{y_0} \in \{10, 00\}.\end{aligned}$$

It can be easily verify that $\rho_0(v, w) = \rho_{\lceil\frac{n+1}{2}\rceil-2}(v, w) = 0$ and $\rho_i(v, w) = \rho_i(x, y) = 1$ for $i \in \{0, 1, \ldots, \lceil\frac{n+1}{2}\rceil - 1\} - \{0, \lceil\frac{n+1}{2}\rceil - 2\}$. By using the CSH algorithm, we can construct a shortest path P', whose length is $\lceil\frac{n+1}{2}\rceil - 2$, between v and w in CQ_n. By the CSH algorithm, for any $z \in P'$, we have

$$\begin{aligned}\text{bit}(z,1)\text{bit}(z,0) &= \text{bit}(u,1)\overline{\text{bit}(u,0)} = \text{bit}(v,1)\text{bit}(v,0)\\ &= \text{bit}(w,1)\text{bit}(w,0) = \overline{x_1x_0} = y_1\overline{y_0} \in \{10, 00\}.\end{aligned}$$

Therefore, $z \notin \{x, y, u\}$. That is, $\{x, y, u\} \bigcap P' = \emptyset$. Then,

$$x, u, P', y$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 3. $x_1x_0 = y_1\overline{y_0}$. If $x_1x_0 \in \{00, 10\}$, then let $y \overset{1}{\text{—}} u \overset{n-2}{\text{—}} v \overset{0}{\text{—}} w$. Otherwise, $x_1x_0 \in \{01, 11\}$, let $x \overset{1}{\text{—}} u' \overset{n-2}{\text{—}} v' \overset{0}{\text{—}} w'$. Similar to Case 2.1, we can construct a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 4. $x_1x_0 = y_1y_0$. Then, $x_1x_0 \in \{01, 11\}, (x_1x_0, y_1y_0) \in \{(01, 01), (11, 11)\}$, and $\sum_{k=1}^{\lceil\frac{n+1}{2}\rceil-1} \rho_k(x, y)$ is odd. Let $x \overset{0}{\text{—}} u$, $y \overset{0}{\text{—}} v$. Then, $\text{bit}(u,1)\text{bit}(u,0) = \text{bit}(v,1)\ \text{bit}(v,0) \in \{00, 10\}$, $\rho_0(u, v) = 0$, and $\text{dist}(CQ_n, u, v) = \lceil\frac{n+1}{2}\rceil - 1$. By using the CSH algorithm, we can get a shortest path P, whose length is $\lceil\frac{n+1}{2}\rceil - 1$, between u and v in CQ_n. By the CSH algorithm, for any $z \in P$, we have $\text{bit}(z,1)\text{bit}(z,0) = \text{bit}(u,1)\text{bit}(u,0) \in \{00, 10\}$ and, thus, $x_1x_0 = y_1y_0 \neq \text{bit}(z,1)\text{bit}(z,0)$. As a sequence, $\{x, y\} \bigcap P = \emptyset$. Then,

$$x, P, y$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n. □

Lemma 4. *If $n \geq 5$ and n is odd, $x = 0x_{n-2}\ldots x_1x_0 \in V(CQ^0_{n-1})$ and $y = 1y_{n-2}\ldots y_1y_0 \in V(CQ^1_{n-1})$ with $\overline{x_{n-2}}x_{n-3} \overset{d.p.}{\sim} y_{n-2}y_{n-3}$ and $x_{n-2}x_{n-3} \overset{d.p.}{\sim} \overline{y_{n-2}}y_{n-3}$, and $\text{dist}(CQ_n, x, y) = \lceil\frac{n+1}{2}\rceil - 1$, then there is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.*

Proof. We separately consider the two cases that $\rho_0(x, y) = 1$ and $\rho_0(x, y) = 0$.

Case 1. $\rho_0(x, y) = 1$. Since $\rho_i(x, y) = 1$ for $i \in \{\lceil\frac{n+1}{2}\rceil - 1, \lceil\frac{n+1}{2}\rceil - 2\}$, $\lceil\frac{n+1}{2}\rceil = \text{diam}(CQ_n) > \text{dist}(CQ_n, x, y) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \rho_k(x, y) \geq 3$. Considering that n is odd, $n \geq 7$. We consider the cases as below.

Case 1.1. $x_1x_0 \in \{00, 10\}$. Then, $y_1y_0 \in \{\overline{x_1}0, x_11, \overline{x_1}1\}$.

Case 1.1.1. $y_1y_0 = \overline{x_1}0$. Let $y \overset{0}{\text{—}} u \overset{1}{\text{—}} v \overset{0}{\text{—}} w$. Then, $\rho_0(x, w) = 0$ and $\rho_i(x, w) = \rho_i(x, y)$ for $i \in \{1, 2, \ldots, \lceil\frac{n+1}{2}\rceil - 1\}$. Thus, $\text{dist}(CQ_n, x, w) = \lceil\frac{n+1}{2}\rceil - 2$. By using the CSH algorithm, we can get a shortest path P between x and w in CQ_n. Similar to the proof in Lemma 3, we can deduce that $\{y, u, v\} \bigcap P = \emptyset$. Then,

$$P, v, u, y$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 1.1.2. $y_1y_0 = x_11$. Let $y \overset{1}{\text{—}} u \overset{0}{\text{—}} v \overset{1}{\text{—}} w$. Then, similar to Case 1.1.1, we can construct a shortest path P' between x and w in CQ_n and by using the path P' we can get a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 1.1.3. $y_1y_0 = \overline{x_1}1$. Let $y \overset{1}{\text{—}} u \overset{n-2}{\text{—}} v \overset{0}{\text{—}} z \overset{1}{\text{—}} w$. Then, $\rho_0(x, w) = \rho_{\lceil\frac{n+1}{2}\rceil-2}(x, w) = 0$ and $\rho_i(x, w) = \rho_i(x, y)$ for $i \in \{0, 1, \ldots, \lceil\frac{n+1}{2}\rceil - 1\} - \{0, \lceil\frac{n+1}{2}\rceil - 2\}$. Thus, $\text{dist}(CQ_n, x, w) = \lceil\frac{n+1}{2}\rceil - 3$. By using CSH algorithm, we can get a shortest path P'' between x and w in CQ_n and we can deduce that $\{y, u, v, z\} \bigcap P'' = \emptyset$. Then,

$$P, z, v, u, y$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 1.2. $x_1x_0 \in \{01, 11\}$. Then, $y_1y_0 \in \{00, 10, 01, 11\}$. In fact, exchanging the position of x and y, the case for $y_1y_0 \in \{00, 10\}$ can be reduced to Case 1.1. So, we only consider the cases for $y_1y_0 \in \{01, 11\}$. For $y_1y_0 \in \{01, 11\}$, let $y \overset{0}{\text{—}} u \overset{1}{\text{—}} v \overset{0}{\text{—}} w$. Similar to the proof in Lemma 3, we can construct a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 2. $\rho_0(x, y) = 0$. Then, $\rho_i(x, y) = 1$ for $i \in \{1, 2, \ldots, \lceil\frac{n+1}{2}\rceil - 1\}$. We deal with the following cases.

Case 2.1. $x_1x_0 \in \{00, 10\}$. Let $y \overset{1}{\text{—}} u \overset{n-2}{\text{—}} v \overset{1}{\text{—}} w$. Then, $\rho_0(x, w) = \rho_{\lceil\frac{n+1}{2}\rceil-2}(x, w) = 0$ and $\rho_i(x, w) = 1$ for $i \in \{0, 1, \ldots, \lceil\frac{n+1}{2}\rceil - 1\} - \{0, \lceil\frac{n+1}{2}\rceil - 2\}$. Thus, $\text{dist}(CQ_n, x, w) = \lceil\frac{n+1}{2}\rceil - 2$. By using the CSH algorithm, we can get a shortest path P between x and w in CQ_n and we can deduce that $\{y, u, v\} \bigcap P = \emptyset$. Then,

$$P, v, u, y$$

is a path of length $\lceil\frac{n+1}{2}\rceil + 1$ between x and y in CQ_n.

Case 2.2. $x_1x_0 \in \{01, 11\}$. Let $x \overset{1}{\text{—}} u$, $y \overset{1}{\text{—}} v$. Then, $\rho_0(u, v) = \rho_0(x, y) = 0$. Let $P' : u = z_0, z_1, \ldots, z_k = v$ be a shortest path got by using CSH algorithm between u and v in CQ_n. Supposing that $z_{i-1} \overset{j_i}{\text{—}} z_i$ for $i \in \{1, 2, \ldots, k\}$, we will

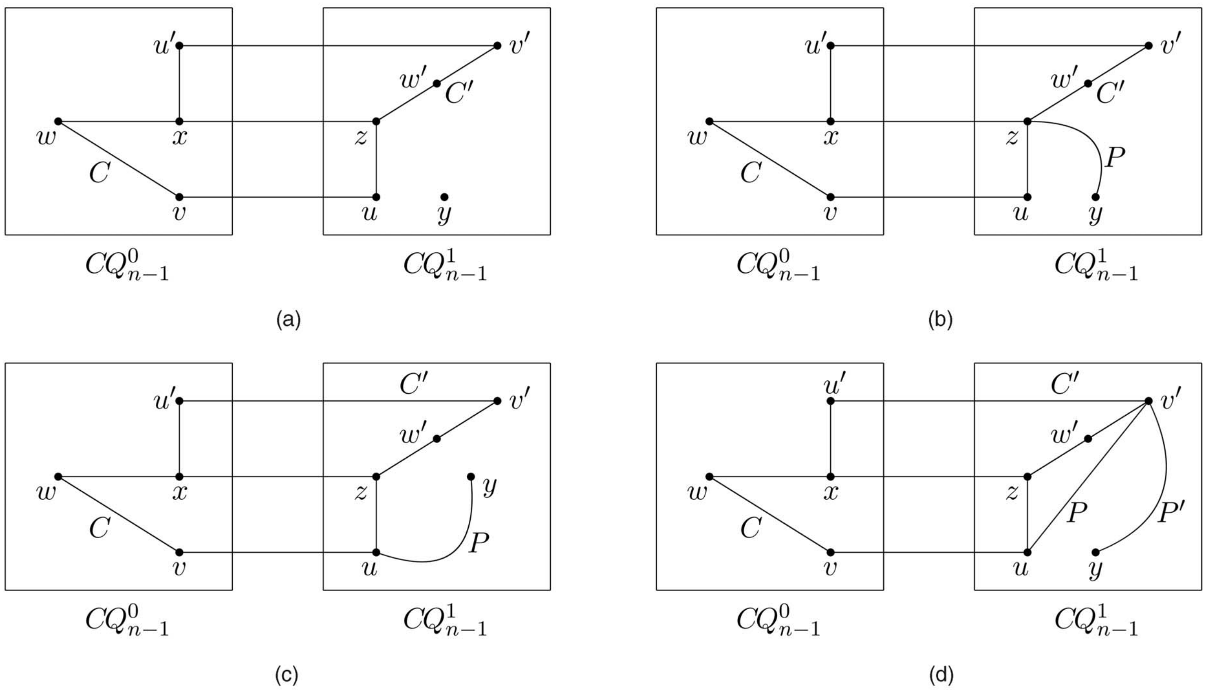

Fig. 2. A path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_τ, where a straight line represents an edge and a curve line represents a path between two nodes.

prove $x \notin P'$. Since $\rho_0(u,v)=0$, by the CSH algorithm, $j_1 \neq 1$ and, thus, $x \neq z_1$. Further, for $2 \leq i \leq k$, by the CSH algorithm, $\mathrm{dist}(CQ_n, u, z_i) = \mathrm{dist}(CQ_n, z_0, z_i) = i \geq 2 > 1 = \mathrm{dist}(CQ_n, z_0, x)$. Therefore, $x \neq z_i$ for $i \in \{2, 3, \ldots, k\}$. And, it is clear that $x \neq z_0 = u$. To sum up, $x \notin \{z_0, z_1, \ldots, z_k\}$. That is, $x \notin P'$. Similarly, we can prove $y \notin P'$. Thus,

$$x, P', y$$

is a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_n. □

The proof of Lemma 7 need to use the following lemma:

Lemma 5. *If $n \geq 3$, $x \stackrel{n-1}{\text{---}} y$ for $x, y \in V(CQ_n)$, letting $x \stackrel{0}{\text{---}} u \stackrel{n-1}{\text{---}} v \stackrel{0}{\text{---}} w \stackrel{1}{\text{---}} z$ and $y \stackrel{0}{\text{---}} u' \stackrel{n-1}{\text{---}} v' \stackrel{0}{\text{---}} w' \stackrel{1}{\text{---}} z'$, then $z = y$, $z' = x$, and $C : x, u, v, w, y, x$ and $C' : y, u', v', w', x, y$ are the two cycles of length 5 that contain the edge (x, y) in CQ_n. Furthermore, $u \notin \{v', w'\}$ and $u' \notin \{v, w\}$.*

Proof. Let $x = 0x_{n-2} \ldots x_1 x_0$, $y = 1y_{n-2} \ldots y_1 y_0$. Then, $(x_1x_0, y_1y_0) \in \{(x_1 0, x_1 0), (x_1 1, \overline{x_1} 1)\}$. By Definition 2, we can verify that $z = y$, $z' = x$, and $C : x, u, v, w, y, x$ and $C' : y, u', v', w', x, y$ are the two cycles of length 5 that contain the edge (x, y) in CQ_n. Further, if $(x_1x_0, y_1y_0) = (x_1 0, x_1 0)$, then $\mathrm{bit}(u,1)\mathrm{bit}(u,0) = x_1 1$, $\mathrm{bit}(v',1)\mathrm{bit}(v',0) = \overline{x_1} 1$, and $\mathrm{bit}(w',1)\mathrm{bit}(w',0) = \overline{x_1} 0$, while if $(x_1x_0, y_1y_0) = (x_1 1, \overline{x_1} 1)$, then $\mathrm{bit}(u,1)\mathrm{bit}(u,0) = x_1 0$, $\mathrm{bit}(v',1)\mathrm{bit}(v',0) = \overline{x_1} 0$, and $\mathrm{bit}(w',1)\mathrm{bit}(w',0) = \overline{x_1} 1$. Thus, $u \notin \{v', w'\}$. Similarly, we can also verify that $u' \notin \{v, w\}$. □

The following lemma will also be used in the proof of Lemma 7 and Theorem 1.

Lemma 6 [15]. *If $n \geq 2$, for any $(x, y) \in E(CQ_n)$ and any integer l with $4 \leq l \leq 2^n$, there exists a cycle C of length l in CQ_n such that (x, y) is in C.*

Lemma 7. *If $n \geq 7$ and n is odd, $x = 0x_{n-2} \ldots x_1 x_0 \in V(CQ^0_{n-1})$ and $y = 1y_{n-2} \ldots y_1 y_0 \in V(CQ^1_{n-1})$ with $\overline{x_{n-2}} x_{n-3} \stackrel{d.p.}{\sim} y_{n-2} y_{n-3}$ and $x_{n-2} x_{n-3} \stackrel{d.p.}{\sim} \overline{y_{n-2}} y_{n-3}$, and $\mathrm{dist}(CQ_n, x, y) = \lceil \frac{n+1}{2} \rceil - 2$, then there is a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_n.*

Proof. Let $x \stackrel{n-1}{\text{---}} z$. Then, $z \in V(CQ^1_{n-1})$, $\rho_{\lceil \frac{n+1}{2} \rceil - 1}(z, y) = 0$, and $\rho_i(z, y) = \rho_i(x, y)$, $i = 0, 1, \ldots, \lceil \frac{n+1}{2} \rceil - 2$. Hence, $\mathrm{dist}(CQ^1_{n-1}, z, y) = \lceil \frac{n+1}{2} \rceil - 3 \geq 1$ and, thus, $y \neq z$. Without loss of generality, we assume that $x \in V(CQ^{000}_{n-3})$ (similarly prove in other cases). Then $z \in V(CQ^{100}_{n-3})$ and $y \in V(CQ^{110}_{n-3})$. Let $z \stackrel{0}{\text{---}} u \stackrel{n-1}{\text{---}} v \stackrel{0}{\text{---}} w \stackrel{1}{\text{---}} x'$ and $x \stackrel{0}{\text{---}} u' \stackrel{n-1}{\text{---}} v' \stackrel{0}{\text{---}} w' \stackrel{1}{\text{---}} z'$. By the conditions in the lemma, we can easily verify that $y_{n-2} = \overline{\mathrm{bit}(u, n-2)} = \overline{\mathrm{bit}(v', n-2)} = \overline{\mathrm{bit}(w', n-2)}$. Therefore, $y \notin \{u, v', w'\}$. By Lemma 5, we have $x' = x$ and $z' = z$ and, thus, $C : z, u, v, w, x, z$ and $C' : x, u', v', w', z, x$ are two cycles of length 5 in CQ_n such that $u \notin \{v', w'\}$ and $u' \notin \{v, w\}$ (see Fig. 2a). Since $\mathrm{dist}(CQ^1_{n-1}, z, y) = \lceil \frac{n+1}{2} \rceil - 3 \geq 1$, we may let P be a path of length $\lceil \frac{n+1}{2} \rceil - 3$ between z and y in CQ^1_{n-1}. Note that P is also a shortest path between z and y in CQ^1_{n-1}. We deal with the following cases.

Case 1. $u \notin P$. Then,

$$x, w, v, u, P$$

is a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_n (see Fig. 2b).

Case 2. $u \in P$. Since $(z, u) \in E(CQ^1_{n-1})$ and P is a shortest path between z and y in CQ^1_{n-1}, P must be: $z, u, \ldots, y$. We can further claim that $w' \notin P$. Otherwise, P must be $z, u, \ldots, w', \ldots, y$. Thus, $\mathrm{len}(\mathrm{path}(P, z, w')) > 1$. On the other hand, since P is a shortest path between z and y in CQ^1_{n-1}, the subpath between z and w' in P must be a shortest path between z and w' in CQ^1_{n-1}.

Considering that $(z, w') \in E(CQ^1_{n-1})$, the length of the subpath between z and w' in P should be 1. So, we obtain a contradiction. Thus, we have the following cases:

Case 2.1. $v' \notin P$. Then,

$$x, u', v', w', P$$

is a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_n (see Fig. 2c).

Case 2.2. $v' \in P$. Then, P is $z, u, \ldots, v', \ldots, y$. Since $(z, v') \notin E(CQ^1_{n-1})$ (Otherwise, z, v', w', z is a cycle of length 3 in CQ^1_{n-1}, but there is no cycle of length 3 in CQ_n for all $n \geq 1$, which can be proved similar to the proof of Lemma 5 in [13]), z, w', v' is a shortest path between z and v' in CQ^1_{n-1}. Thus, $\text{dist}(CQ^1_{n-1}, z, v') = 2$. Considering that P is a shortest path between z and y in CQ^1_{n-1}, the subpath between z and v' in P must be a shortest path between z and v' in CQ^1_{n-1}. Therefore, $\text{len}(\text{path}(P, z, v')) = 2$. Since $(z, u) \in E(CQ^1_{n-1})$, $(u, v') \in V(CQ^1_{n-1})$ (See Fig. 2d). Let $P' = \text{path}(P, v', y)$. Then, $\text{len}(P') = \lceil \frac{n+1}{2} \rceil - 5$. Thus,

$$x, w, v, u, z, w', P'$$

is a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between x and y in CQ_n. □

Further, Lemma 8 shows a property on distance-preserving pair related, which will be used in the proof of Theorem 1.

Lemma 8. *If $n \geq 3$ and n is odd, $x = 0x_{n-2} \ldots x_1 x_0 \in V(CQ^0_{n-1})$ and $y = 1y_{n-2} \ldots y_1 y_0 \in V(CQ^1_{n-1})$, $u \overset{n-1}{\rule{1em}{0.4pt}} x$, and $v \overset{n-1}{\rule{1em}{0.4pt}} y$ such that $\text{bit}(v, n-2) \neq \text{bit}(x, n-2)$ and $\text{bit}(u, n-2) \neq \text{bit}(y, n-2)$, then $\overline{x_{n-2}} x_{n-3} \overset{d.p.}{\sim} y_{n-2} y_{n-3}$ and $x_{n-2} x_{n-3} \overset{d.p.}{\sim} \overline{y_{n-2}} y_{n-3}$.*

Proof. According to the definition of function ρ, we can easily claim that $(x_{n-2}x_{n-3}, y_{n-2}y_{n-3}) \in \{(00, 10), (10, 00), (01, 01), (11, 11)\}$. We can easily verify $\overline{x_{n-2}} x_{n-3} \overset{d.p.}{\sim} y_{n-2} y_{n-3}$ and $x_{n-2} x_{n-3} \overset{d.p.}{\sim} \overline{y_{n-2}} y_{n-3}$. □

So far, we have introduced a series of lemmas. Now, let us give a proof that there exists paths of all lengths from $\lceil \frac{n+1}{2} \rceil + 1$ through $2^n - 1$ between any two distinct nodes in CQ_n for $n \geq 3$. Even with the the help of these lemmas, we point out, we still have to deal with many cases in this proof.

Theorem 1. *If $n \geq 3$, for any $x, y \in V(CQ_n)$, $x \neq y$, and any integer l, $\lceil \frac{n+1}{2} \rceil + 1 \leq l \leq 2^n - 1$, there exists a path of length l between x and y in CQ_n.*

Proof. we use induction on n.

We can easily verify the truth of the theorem for $n = 3$. Suppose that the theorem holds for $3 \leq n \leq \tau - 1$, we will prove that the theorem holds for $n = \tau$ ($\tau \geq 4$).

When $n = \tau$, for any $x, y \in V(CQ_\tau)$, $x \neq y$, and any integer l with $\lceil \frac{\tau+1}{2} \rceil + 1 \leq l \leq 2^\tau - 1$, without loss generality, we separately identify the two cases that both x and y are in $CQ^0_{\tau-1}$, and x and y are, respectively, in $CQ^0_{\tau-1}$ and $CQ^1_{\tau-1}$ to construct a path of length l in CQ_τ.

First, we consider the case that both x and y are in $CQ^0_{\tau-1}$. In this case, we further deal with the following cases according to the value of l:

Case 1. $\lceil \frac{\tau+1}{2} \rceil + 1 \leq l \leq 2^{\tau-1} - 1$. By the induction hypothesis, there exist paths of lengths $\lceil \frac{\tau}{2} \rceil + 1$, $\lceil \frac{\tau}{2} \rceil + 2$, $\ldots$, $2^{\tau-1} - 1$ between x and y in $CQ^0_{\tau-1}$. Noticing that $\lceil \frac{\tau}{2} \rceil + 1 \leq \lceil \frac{\tau+1}{2} \rceil + 1$, there exist paths of lengths l between x and y in $CQ^0_{\tau-1}$ and, thus, in CQ_τ.

Case 2. $2^{\tau-1} \leq l \leq 2^\tau - 1$. Let $l_1 = \lceil \frac{l-1}{2} \rceil$, $l_2 = \lfloor \frac{l-1}{2} \rfloor$. Then, $\lceil \frac{\tau}{2} \rceil + 1 \leq 2^{\tau-2} - 1 \leq l_2 \leq l_1 \leq 2^{\tau-1} - 1$ and $l_1 + l_2 = l - 1$. By the induction hypothesis, there exists a path P_1 of length l_1 between x and y in $CQ^0_{\tau-1}$. Select an edge (u, v) in P_1 (see Fig. 3a). Let $u' \overset{\tau-1}{\rule{1em}{0.4pt}} u$ and $v' \overset{\tau-1}{\rule{1em}{0.4pt}} v$. Then, $u', v' \in V(CQ^1_{\tau-1})$ and $u' \neq v'$. By the induction hypothesis, there exists a path P_2 of length l_2 between u' and v' in $CQ^1_{\tau-1}$. Then,

$$\text{path}(P_1, x, u), P_2, \text{path}(P_1, v, y)$$

is a path of length $l_1 + l_2 + 1 = l$ between x and y in CQ_τ.

Next, we consider the case that x in $CQ^0_{\tau-1}$ and y in $CQ^1_{\tau-1}$. In this case, we further deal with two cases according to whether x and y are adjacent as follows:

Case 1. $(x, y) \in E(CQ_\tau)$. By Lemma 6, for any l' with $4 \leq l' \leq 2^\tau$, there exists a cycle C of length l' that contains the edge (x, y) in CQ_τ. So, for any l'' with $3 \leq l'' \leq 2^\tau - 1$, there is a path $C - (x, y)$ of length l'' between x and y in CQ_τ. Since $\tau \geq 4$, $\lceil \frac{\tau+1}{2} \rceil + 1 \geq 4$. Thus, there exists a path of length l with $\lceil \frac{\tau+1}{2} \rceil + 1 \leq l \leq 2^\tau - 1$ between x and y in CQ_τ.

Case 2. $(x, y) \notin E(CQ_\tau)$. We have the following cases according to the range of l.

Case 2.1. $\lceil \frac{\tau+1}{2} \rceil + 1 \leq l \leq 2^{\tau-1}$. Let $z \overset{\tau-1}{\rule{1em}{0.4pt}} x$. Then, $z \in V(CQ^1_{\tau-1})$ and $z \neq y$. Let $l' = l - 1$. We deal with the following cases:

Case 2.1.1. τ is even. Then, $\lceil \frac{\tau}{2} \rceil + 1 = \lceil \frac{\tau+1}{2} \rceil \leq l' \leq 2^{\tau-1} - 1$. By the induction hypothesis, there exists a path P of length l' between z and y in $CQ^1_{\tau-1}$. Thus,

$$x, P$$

is a path of length $l' + 1 = l$ with $\lceil \frac{\tau+1}{2} \rceil + 1 \leq l \leq 2^{\tau-1}$ between x and y in CQ_τ.

Case 2.1.2. τ is odd. Then, $\tau \geq 5$. For $\lceil \frac{\tau+1}{2} \rceil + 2 \leq l \leq 2^{\tau-1}$, $\lceil \frac{\tau}{2} \rceil + 1 = \lceil \frac{\tau+1}{2} \rceil + 1 \leq l' \leq 2^{\tau-1} - 1$. Similar to Case 2.1.1, we can get a path of length $l' + 1 = l$ between x and y in CQ_τ. In order to show that there exists a path of length $l = \lceil \frac{\tau+1}{2} \rceil + 1$ between x and y in CQ_τ, let $v \overset{\tau-1}{\rule{1em}{0.4pt}} y$. Then, $v \in V(CQ^0_{\tau-1})$ and $v \neq x$. We consider the following cases:

Case 2.1.2.1. $\text{bit}(v, \tau-2) = \text{bit}(x, \tau-2)$ or $\text{bit}(z, \tau-2) = \text{bit}(y, \tau-2)$. Then, there exists an $i \in \{0, 1\}$ such that $\{v, x\} \subset V(CQ^{0i}_{\tau-2})$ or $\{z, y\} \subset V(CQ^{1i}_{\tau-2})$. Without loss of generality, we assume that there exists an $i \in \{0, 1\}$ such that $\{v, x\} \subset V(CQ^{0i}_{\tau-2})$. By the induction hypothesis, there exists a path P of length $\lceil \frac{\tau-1}{2} \rceil + 1$ between x and v in $CQ^{0i}_{\tau-2}$. Thus,

$$P, y$$

is a path of length $\lceil \frac{\tau-1}{2} \rceil + 2 = \lceil \frac{\tau+1}{2} \rceil + 1$ between x and y in CQ_τ.

Case 2.1.2.2. $\text{bit}(v, \tau-2) \neq \text{bit}(x, \tau-2)$ and $\text{bit}(z, \tau-2) \neq \text{bit}(y, \tau-2)$. By Lemma 8, we have $\overline{x_{\tau-2}} x_{\tau-3} \overset{d.p}{\sim} y_{\tau-2} y_{\tau-3}$ and $x_{\tau-2} x_{\tau-3} \overset{d.p.}{\sim} \overline{y_{\tau-2}} y_{\tau-3}$. Considering that $\text{diam}(CQ_\tau) = \lceil \frac{\tau+1}{2} \rceil$ for $\tau \geq 1$, we deal with the following cases:

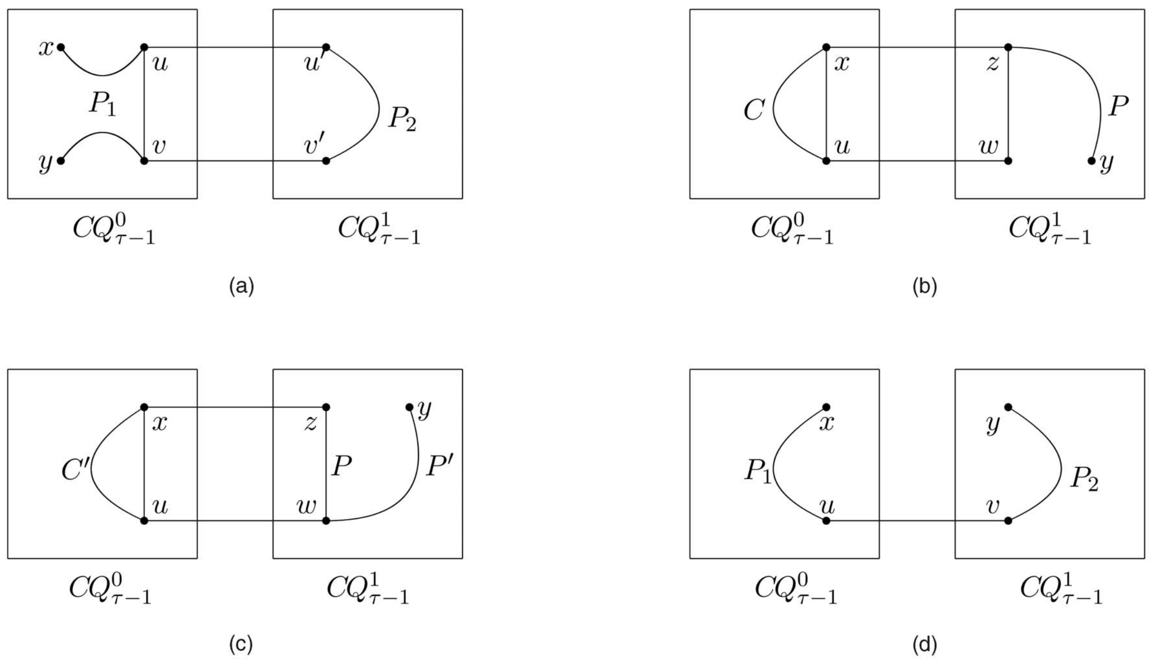

Fig. 3. A path of length l between x and y in CQ_τ, where a straight line represents an edge and a curve line represents a path between two nodes.

Case 2.1.2.2.1. $\lceil\frac{\tau+1}{2}\rceil - 2 \le \mathrm{dist}(CQ_\tau, x, y) \le \lceil\frac{\tau+1}{2}\rceil$. If $\mathrm{dist}(CQ_\tau, x, y) = \lceil\frac{\tau+1}{2}\rceil - 2$, we can deduce $\tau \ne 5$. Supposing on the contrary that $\tau = 5$, $\mathrm{dist}(CQ_\tau, x, y) = \lceil\frac{5+1}{2}\rceil - 2 = 1$. Hence, $(x, y) \in E(CQ_\tau)$, contradictory to $(x, y) \notin E(CQ_\tau)$. So, we have $\tau \ge 7$ if $\mathrm{dist}(CQ_\tau, x, y) = \lceil\frac{\tau+1}{2}\rceil - 2$. By Lemma 7, there exists a path of length $\lceil\frac{\tau+1}{2}\rceil + 1$ between x and y in CQ_τ for $\mathrm{dist}(CQ_\tau, x, y) = \lceil\frac{\tau+1}{2}\rceil - 2$. For $(CQ_\tau, x, y) \in \{\lceil\frac{\tau+1}{2}\rceil, \lceil\frac{\tau+1}{2}\rceil - 1\}$, by Lemma 3 and Lemma 4, there exists a path of length $\lceil\frac{\tau+1}{2}\rceil + 1$ between x and y in CQ_τ.

Case 2.1.2.2.2. $2 \le \mathrm{dist}(CQ_\tau, x, y) \le \lceil\frac{\tau+1}{2}\rceil - 3$. Notice that $\lfloor\frac{\tau-1}{2}\rfloor = \lceil\frac{\tau+1}{2}\rceil - 1$. Since $z \overset{\tau-1}{\sim} x$, we have $\rho_{\lceil\frac{\tau+1}{2}\rceil-1}(z, y) = 0$, $\rho_{\lceil\frac{\tau+1}{2}\rceil-2}(z, y) = \rho_{\lceil\frac{\tau+1}{2}\rceil-2}(x, y) = 1$. Therefore, $\rho_i(z, y) = \rho_i(x, y)$ for $i = 0, 1, \ldots, \lceil\frac{\tau+1}{2}\rceil - 3$ and $1 \le \mathrm{dist}(CQ^1_{\tau-1}, z, y) \le \lceil\frac{\tau+1}{2}\rceil - 4$. Let P be a shortest path obtained using the CSH algorithm between z and y in $CQ^1_{\tau-1}$. Then, $\mathrm{len}(P) = \mathrm{dist}(CQ^1_{\tau-1}, z, y)$. By Lemma 2, there exists a cycle x, u, w, z, x of length 4 that contains the edge (x, z) in CQ_τ. The following proof will separately discuss according to whether $w \in P$.

If $w \notin P$. Let $m = \lceil\frac{\tau+1}{2}\rceil - \mathrm{dist}(CQ^1_{\tau-1}, z, y)$. Then, $4 \le m \le \lceil\frac{\tau+1}{2}\rceil - 1 \le 2^{\tau-1} - 1$. By Lemma 6, there exists a cycle C of length m that contains the edge (x, u) in $CQ^0_{\tau-1}$ (see Fig. 3b). Thus,

$$C - (x, u), w, P$$

is a path of length $(m - 1) + 2 + \mathrm{dist}(CQ^1_{\tau-1}, z, y) = \lceil\frac{\tau+1}{2}\rceil + 1$ between x and y in CQ_τ.

Otherwise, since P is a shortest path between z and y in $CQ^1_{\tau-1}$, $\mathrm{path}(P, z, w)$ is a shortest path in $CQ^1_{\tau-1}$. And, considering that $(z, w) \in V(CQ^1_{\tau-1})$, P must be $z, w, \ldots, y$. Let $P' = \mathrm{path}(P, w, y)$. Then, $\mathrm{len}(P') = \mathrm{dist}(CQ^1_{\tau-1}, z, y) - 1$. Let $m' = \lceil\frac{\tau+1}{2}\rceil - \mathrm{dist}(CQ^1_{\tau-1}, z, y) + 2$. Then, $6 \le m' \le \lceil\frac{\tau+1}{2}\rceil + 1 \le 2^{\tau-1} - 1$. By Lemma 6, there exists a cycle C' of length m' that contains the edge (x, u) in $CQ^0_{\tau-1}$ (see Fig. 3c). Thus,

$$C' - (x, u), P'$$

is a path of length $(m' - 1) + 1 + (\mathrm{dist}(CQ^1_{\tau-1}, z, y) - 1) = \lceil\frac{\tau+1}{2}\rceil + 1$ between x and y in CQ_τ.

Case 2.2. $2^{\tau-1} + 1 \le l \le 2^\tau - 1$. By Definition 2, we can select $u \in V(CQ^0_{\tau-1}) - \{x\}$, $v \in V(CQ^1_{\tau-1}) - \{y\}$ such that $(u, v) \in E(CQ_\tau)$. Let $l_1 = \lceil\frac{l-1}{2}\rceil$, $l_2 = \lfloor\frac{l-1}{2}\rfloor$. Then, $\lceil\frac{\tau}{2}\rceil + 1 \le 2^{\tau-2} \le l_2 \le l_1 \le 2^{\tau-1} - 1$ and $l_1 + l_2 = l - 1$. By the induction hypothesis, there exist a path P_1 of length l_1 between x and u in $CQ^0_{\tau-1}$ and a path P_2 of length l_2 between v and y in $CQ^1_{\tau-1}$ (see Fig. 3d). Then,

$$P_1, P_2$$

is a path of length $l_1 + l_2 + 1 = l$ between x and y in CQ_τ. □

The similar result was obtained independently by Yang et al. in [35]. However, the result in [35] deduces the existence of paths of lengths greater than or equal to $\lceil\frac{n+1}{2}\rceil + 2$, but does not include $\lceil\frac{n+1}{2}\rceil + 1$.

4 NONEXISTENCE OF EMBEDDING PATHS OF LENGTHS $\lceil\frac{n+1}{2}\rceil$

In Section 3, we have given the embedding of paths of length from $\lceil\frac{n+1}{2}\rceil + 1$ through $2^n - 1$ between any two distinct nodes with a dilation of 1 in CQ_n. The following theorem will prove that $\lceil\frac{n+1}{2}\rceil + 1$ is the shortest possible length that can be embedded between arbitrary two distinct nodes with a dilation of 1 in CQ_n.

Theorem 2. *If $n \ge 2$, then there exists two nodes $x, y \in V(CQ_n)$ such that $x \ne y$ and there is no path of length $\lceil\frac{n+1}{2}\rceil$ between x and y in CQ_n.*

Proof. The theorem obviously holds for $n = 2$. Selecting $x = 001$, $y = 111$ for $n = 3$ and selecting $x = 0001$, $y = 1101$ for $n = 4$, we can easily verify the truth of the theorem for $n = 3, 4$ also. For $n \geq 5$, we separately deal with two cases as follows:

Case 1. $\lceil \frac{n+1}{2} \rceil$ is even. Select $x = x_{n-1}x_{n-2}\ldots x_1x_0 = 0^{n-1}1$, $y = y_{n-1}y_{n-2}\ldots y_1y_0 = 1^n$. Then, $\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x, y) = \lceil \frac{n+1}{2} \rceil - 1$ is odd. By the definition of function ρ, $\rho_0(x, y) = 0$.

Suppose that there exists a path P of length $\lceil \frac{n+1}{2} \rceil$ between x and y in CQ_n. Let P be $x = x^{(0)}, x^{(1)}, \ldots, x^{(\lceil \frac{n+1}{2} \rceil)} = y$, where $x^{(k)} = x^{(k)}_{n-1}x^{(k)}_{n-2}\ldots x^{(k)}_1 x^{(k)}_0 k = 0, 1,$ for $\ldots, \lceil \frac{n+1}{2} \rceil$, and $x^{(0)} \overset{j_1}{-} x^{(1)}$. We can claim that $j_1 \notin \{0, 1\}$. Supposing on the contrary that $j_1 \in \{0, 1\}$, then we have $\rho_0(x^{(1)}, y) = 1$ and $\rho_i(x^{(1)}, y) = \rho_i(x, y)$ for $i = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and, thus, $\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(1)}, y) = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x, y) = \lceil \frac{n+1}{2} \rceil - 1$. So, the length of P is greater than

$$\mathrm{dist}(CQ_n, x^{(1)}, y) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(1)}, y) = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(1)}, y) + \rho_0(x^{(1)}, y) = \left\lceil \frac{n+1}{2} \right\rceil,$$

contradictory to that the length of P is $\lceil \frac{n+1}{2} \rceil$.

Further, if n is odd, then $j_1 \neq n-1$. Otherwise, we have $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(1)}, y) = 0$, $\rho_{\lfloor \frac{n-1}{2} \rfloor - 1}(x^{(1)}, y) = 2$, and $\rho_i(x^{(1)}, y) = 1$ for $i = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 2$. Similar to the above discussion, we can deduce that the length of P is greater than $\lceil \frac{n+1}{2} \rceil$, a contradiction.

Moreover, if $j_1 \in \{2, 3, \ldots, 2\lfloor \frac{n-1}{2} \rfloor - 1\}$, we can claim that j_1 is not odd and if j_1 is even, then $\sum_{i=\frac{j_1}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(0)}, y)$ is odd. Otherwise, we have $\rho_i(x^{(1)}, y) = \rho_i(x, y)$ for $i = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and $\rho_0(x^{(1)}, y) = 1$. Thus, we can get the same contradiction as the above discussion.

In summary, j_1 satisfies one of the two conditions as follows:

1. j_1 is even with $2 \leq j_1 \leq 2\lfloor \frac{n-1}{2} \rfloor - 1$ and $\sum_{i=\frac{j_1}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(0)}, y)$ is odd;
2. $j_1 \in \{n-1, n-2\}$ and n is even.

By the definition of function ρ, we have the following conclusions:

a. If j_1 satisfies the condition 1, then $\rho_{\frac{j_1}{2}}(x^{(1)}, y) = \rho_0(x^{(1)}, y) = 0$, $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(1)}, y) = 2$ with $x^{(1)}_{n-1}x^{(1)}_{n-2} = 00$ if n is even, $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(1)}, y) = 1$ with $x^{(1)}_{n-1} = 0$ if n is odd, and $\rho_i(x^{(1)}, y) = 1$ with $x^{(1)}_{2i+1}x^{(1)}_{2i} = 00$ for $i \in \{0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0, \frac{j_1}{2}\}$.

b. If j_1 satisfies the condition 2, then $\rho_0(x^{(1)}, y) = 0$, $\rho_{\lfloor \frac{j_1}{2} \rfloor}(x^{(1)}, y) = 1$ with $x^{(1)}_{n-1}x^{(1)}_{n-2} \in \{01, 10\}$, and $\rho_i(x^{(1)}, y) = 1$ with $x^{(1)}_{2i+1}x^{(1)}_{2i} = 00$ for $i \in \{0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0\}$.

Now, we separately deal with the following cases:

Case 1.1. n is odd. Then, $\lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2} = \lceil \frac{n+1}{2} \rceil - 1$. We assume that $\rho_k(x^{(m)}, y) = 0$ for $k = 0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}$ with

$$\begin{array}{lcl} x^{(m)} & = & \underbrace{0}_{\lfloor \frac{n-1}{2} \rfloor}\ 01\ 01\ldots01\ \underbrace{00}_{k'}\ 00\ldots\ 00\ \underbrace{01}_{k''}\ldots\ldots \\ x^{(m+1)} & = & \underbrace{1}_{\lfloor \frac{n-1}{2} \rfloor}\ 11\ 11\ldots11\ \underbrace{00}_{k'}\ 00\ldots\ 00\ \underbrace{11}_{k''}\ldots\ldots \end{array}$$

Fig. 4. $k' = \max\{i < \lfloor \frac{n-1}{2} \rfloor | \rho_i(x^{(m)}, y) = 1\}$ and $k'' = \max\{i < k' | \rho_i(x^{(m)}, y) = 0\}$, where $\sum_{i=k''+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is odd.

$$1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor - 1$$

and for the even integers j_l with $2 \leq j_l \leq n-2$ and $1 \leq l \leq m$, $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(m)}, y) = 1$ with $x^{(m)}_{n-1} = 0$, and $\rho_{i'}(x^{(m)}, y) = 1$ with $x^{(m)}_{2i'+1}x^{(m)}_{2i'} = 00$ for $i' \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$. Let $x^{(m)} \overset{j_{m+1}}{-} x^{(m+1)}$, $A = \{0, 1, j_1, j_1+1, j_2, j_2+1, \ldots, j_m, j_m+1\}$ and $B = \{j | j$ is odd with $2 \leq j \leq n-2$ and $\rho_{\frac{j-1}{2}}(x^{(m)}, y) = 1\} \bigcup \{j | j$ is even with $2 \leq j \leq n-2$, $\rho_{\frac{j}{2}}(x^{(m)}, y) = 1$, and $\sum_{i=\frac{j}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is even$\}$. We will prove that $j_{m+1} \notin A \bigcup B \bigcup \{n-1\}$ as follows:

Supposing on the contrary that $j_{m+1} \in A \bigcup B \bigcup \{n-1\}$, we have the following cases:

Case 1.1.1. $j_{m+1} \in A$. By the definition of function ρ, we have $\rho_{\lfloor \frac{j_{m+1}}{2} \rfloor}(x^{(m+1)}, y) = 1$, $\rho_{i'}(x^{(m+1)}, y) = 1$ for $i' \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$, and $\rho_k(x^{(m+1)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\} - \{\lfloor \frac{j_{m+1}}{2} \rfloor\}$. Then, $\mathrm{dist}(CQ_n, x^{(m+1)}, y) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m+1)}, y) = \lceil \frac{n+1}{2} \rceil - m$. Thus, the length of P is greater than or equal to $\mathrm{len}(\mathrm{path}(P, x^{(0)}, x^{(m+1)}))$ +dist $(CQ_n, x^{(m+1)}, y) = (m+1) + (\lceil \frac{n+1}{2} \rceil - m) = \lceil \frac{n+1}{2} \rceil + 1$, contradictory to that the length of P is $\lceil \frac{n+1}{2} \rceil$.

Case 1.1.2. $j_{m+1} \in B$. By the definition of function ρ, we have $\rho_{i'}(x^{(m+1)}, y) = 1$ for $i' \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$. Let $k' = \max\{i | \rho_i(x^{(m)}, y) = 0$, and $0 \leq i < \lfloor \frac{j_{m+1}}{2} \rfloor\}$. Then, $k' \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$ and $\sum_{i=k'+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m+1)}, y) = \sum_{i=k'+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$. Thus, $\rho_{k'}(x^{(m+1)}, y) = 1$ and $\rho_k(x^{(m+1)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\} - \{k'\}$. Similar to Case 1.1.1, we can conclude that the length of P is greater than or equal to $\lceil \frac{n+1}{2} \rceil + 1$, contradictory to that the length of P is $\lceil \frac{n+1}{2} \rceil$.

Case 1.1.3. $j_{m+1} = n-1$. Let $k' = \max\{i < \lfloor \frac{n-1}{2} \rfloor | \rho_i(x^{(m)}, y) = 1\}$ (see Fig. 4). Then, $x^{(m)}_{2k'+1}x^{(m)}_{2k'} = 00$ and $\rho_k(x^{(m+1)}, y) = \rho_k(x^{(m)}, y) = 0$ with $x^{(m)}_{2k+1}x^{(m)}_{2k} = 01$ for $k = k'+1, k'+2, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1$ and, thus, $\rho_{k'}(x^{(m+1)}, y) = 2$. Further, let $k'' = \max\{i < k' | \rho_i(x^{(m)}, y) = 0\}$. Then, $\rho_{i'}(x^{(m+1)}, y) = \rho_{i'}(x^{(m)}, y) = 1$ with $x^{(m)}_{2i'+1}x^{(m)}_{2i'} = 00$ for $i' = k''+1, k''+2, \ldots, k'-1$ and, thus, $\sum_{i=k''+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m+1)}, y) = \sum_{i=k''+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$. Obviously, if $\sum_{i=k''+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is odd, then $x^{(m)}_{2k''+1}x^{(m)}_{2k''} = 01$; otherwise, $x^{(m)}_{2k''+1}x^{(m)}_{2k''} = 11$. Without loss of generality, supposing that $\sum_{i=k''+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is odd, then $x^{(m)}_{2k''+1}x^{(m)}_{2k''} = 01$. So, $\rho_{k''}(x^{(m+1)}, y) = 1$. Further, we have $\rho_i(x^{(m+1)}, y) = \rho_i(x^{(m)}, y)$ for $i = 0, 1, \ldots, k''-1$. Similar to the discussion in Case 1.1.1, the length of P is greater than or equal to $\text{len}(P, x^{(0)}, x^{(m+1)}) + \text{dist}(CQ_n, x^{(m+1)}, y) = (m+1) + (\lceil \frac{n+1}{2} \rceil - m) = \lceil \frac{n+1}{2} \rceil + 1$, contradictory to that the length of P is $\lceil \frac{n+1}{2} \rceil$.

To sum up, j_{m+1} is such an even integer that $2 \le j_{m+1} \le n-2$ and $\sum_{i=\frac{j_{m+1}}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is odd with $x^{(m)}_{j_{m+1}+1} x^{(m)}_{j_{m+1}} = 00$. According to the definition of function ρ, we have $\rho_k(x^{(m+1)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_{m+1}}{2}\}$, $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(m+1)}, y) = 1$ with $x^{(m+1)}_{n-1} = 0$, and $\rho_i(x^{(m+1)}, y) = \rho_i(x^{(m)}, y) = 1$ with $x^{(m+1)}_{2i+1} x^{(m+1)}_{2i} = 00$ for $i \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_{m+1}}{2}\}$.

According to the above discussion, we have $x^{(\lfloor \frac{n-1}{2} \rfloor - 1)} = 0(01)^{\lfloor \frac{n-1}{2} \rfloor}$ and, thus, $(x^{(\lfloor \frac{n-1}{2} \rfloor - 1)}, y) \in E(CQ_n)$. Noticing that $\lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil - 1$, we have $x^{(\lceil \frac{n+1}{2} \rceil - 2)}, y) \in E(CQ_n)$. Further, since $(x^{(\lceil \frac{n+1}{2} \rceil - 2)}, x^{(\lceil \frac{n+1}{2} \rceil - 1)}) \in P$ and $x^{(\lceil \frac{n+1}{2} \rceil - 1)}, y) \in P$, there exists a cycle y, $x^{(\lceil \frac{n+1}{2} \rceil - 2)}$, $x^{(\lceil \frac{n+1}{2} \rceil - 1)}$, y in CQ_n, whose length is 3, which contradicts to that there is no cycle of length 3 in CQ_n (the method of proof is similar to that of proof for Lemma 5 in [13]).

Case 1.2. n is even. Then, $\lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil - 2$. We assume that one of the following two conditions holds:

1. $\rho_k(x^{(m)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$ with $1 \le m \le \lfloor \frac{n-1}{2} \rfloor - 1$ and the even integers j_l with $2 \le j_l \le n-3$ and $1 \le l \le m$ and $l \ne k'$, $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(m)}, y) = 2$ with $x^{(m)}_{n-1} x^{(m)}_{n-2} = 00$, and $\rho_{i'}(x^{(m)}, y) = 1$ with $x^{(m)}_{2i'+1} x^{(m)}_{2i'} = 00$ for $i' \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$.
2. $\rho_k(x^{(m)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_{k'-1}}{2}, \frac{j_{k'+1}}{2}, \frac{j_{k'+2}}{2}, \ldots, \frac{j_m}{2}\}$ with $1 \le m \le \lfloor \frac{n-1}{2} \rfloor - 1$ and the even integers j_k with $2 \le j_k \le n-3$, $\rho_{\lfloor \frac{j_{k'}}{2} \rfloor}(x^{(m)}, y) = 1$ with $j_{k'} \in \{n-1, n-2\}$ and $x^{(m)}_{n-1} x^{(m)}_{n-2} \in \{10, 01\}$, and $\rho_{i'}(x^{(m)}, y) = 1$ with $x^{(m)}_{2i'+1} x^{(m)}_{2i'} = 00$ for
$$i' \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \left\{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_{k'-1}}{2}, \frac{j_{k'+1}}{2}, \frac{j_{k'+2}}{2}, \ldots, \frac{j_m}{2}\right\},$$
where $1 \le k' \le m$.

Let $x^{(m)} \xrightarrow{j_{m+1}} x^{(m+1)}$. We deal with the following cases.

Case 1.2.1. Case 1 holds. Let $A' = \{0, 1, j_1, j_1+1, j_2, j_2+1, \ldots, j_m, j_m+1\}$ and $B' = \{j | j$ is odd with $2 \le j \le n-3$ and $\rho_{\frac{j-1}{2}}(x^{(m)}, y) = 1\} \bigcup \{j | j$ is even with $2 \le j \le n-3$, $\rho_{\frac{j}{2}}(x^{(m)}, y) = 1$, and $\sum_{i=\frac{j}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is even$\}$. Then, similar to the discussion in Case 1.1.1 and Case 1.1.2, we have $j_{m+1} \notin A' \bigcup B'$. So, j_{m+1} satisfies one of the following two cases.

Case 1.2.1.1. j_{m+1} is such an even integer that $2 \le j_{m+1} \le n-3$ and $\sum_{i=\frac{j_{m+1}}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is odd with $x^{(m)}_{j_m+1} x^{(m)}_{j_m} = 00$. Then, we have $\rho_k(x^{(m+1)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_{m+1}}{2}\}$ and $\rho_i(x^{(m+1)}, y) = \rho_i(x^{(m)}, y)$ with
$$x^{(m+1)}_{2i+1} x^{(m+1)}_{2i} = x^{(m)}_{2i+1} x^{(m)}_{2i} = 00$$
for $i \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_{m+1}}{2}\}$.

Case 1.2.1.2. $j_{m+1} \in \{n-1, n-2\}$. Then, we have $\rho_{\lfloor \frac{j_{m+1}}{2} \rfloor}(x^{(m+1)}, y) = 1$ with $x^{(m+1)}_{n-1} x^{(m+1)}_{n-2} \in \{10, 01\}$, $\rho_k(x^{(m+1)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$, and $\rho_i(x^{(m+1)}, y) = \rho_i(x^{(m)}, y)$ with $x^{(m+1)}_{2i+1} x^{(m+1)}_{2i} = x^{(m)}_{2i+1} x^{(m)}_{2i} = 00$ for $i \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \ldots, \frac{j_m}{2}\}$.

Case 1.2.2. Case 2 holds. Let $A' = \{0, 1, j_1, j_1+1, \ldots, j_{k'-1}, j_{k'-1}+1, j_{k'+1}, j_{k'+1}+1, j_{k'+2}, j_{k'+2}+1, \ldots, j_m, j_m+1\}$ and $B' = \{j | j$ is odd with $2 \le j \le n-3$ and $\rho_{\frac{j-1}{2}}(x^{(m)}, y) = 1\} \bigcup \{j | j$ is even with $2 \le j \le n-3$, $\rho_{\frac{j}{2}}(x^{(m)}, y) = 1$, and $\sum_{i=\frac{j}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is even $\}$. Then, similar to the discussion in Case 1.1.1 and Case 1.1.2, we have $j_{m+1} \notin A' \bigcup B'$. Further, we can show that $j_{m+1} \notin \{n-1, n-2\}$. Supposing on the contrary that $j_{m+1} \in \{n-1, n-2\}$, we have $x^{(m+1)}_{n-1} x^{(m+1)}_{n-2} \in \{00, 11\}$. If $x^{(m+1)}_{n-1} x^{(m+1)}_{n-2} = 11$, similar to the discuss in Case 1.1.3, we can obtain a contradiction. If $x^{(m+1)}_{n-1} x^{(m+1)}_{n-2} = 00$, we have $\rho_{\lfloor \frac{n-1}{2} \rfloor}(x^{(m+1)}, y) = 2$ and $\rho_i(x^{(m+1)}, y) = \rho_i(x^{(m)}, y)$ for $i = 0, 1 \ldots, \lfloor \frac{n-1}{2} \rfloor - 1$. Hence,
$$\text{dist}(CQ_n, x^{(m+1)}, y) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m+1)}, y) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y) + 1 = \lceil \frac{n+1}{2} \rceil - m.$$

As a result, the length of P is larger than or equal to
$$\text{len}(P, x^{(0)}, x^{(m+1)}) + \text{dist}(CQ_n, x^{(m+1)}, y) = (m+1) + (\lceil \frac{n+1}{2} \rceil - m) = \lceil \frac{n+1}{2} \rceil + 1,$$
contradictory to that the length of P is $\lceil \frac{n+1}{2} \rceil$. Then, j_{m+1} must be such an even integer that $2 \le j_{m+1} \le n-3$ and $\sum_{i=\frac{j_{m+1}}{2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i(x^{(m)}, y)$ is odd with $x^{(m)}_{j_m+1} x^{(m)}_{j_m} = 00$. Further, we have $\rho_k(x^{(m+1)}, y) = 0$ for $k \in \{0, \frac{j_1}{2}, \frac{j_2}{2}, \frac{j_{k'-1}}{2}, \frac{j_{k'+1}}{2}, \frac{j_{k'+2}}{2}, \ldots, \frac{j_{m+1}}{2}\}$, $\rho_{\lfloor \frac{j_{k'}}{2} \rfloor}(x^{(m+1)}, y) = 1$ with $j_{k'} \in \{n-1, n-2\}$ and $x^{(m+1)}_{n-1} x^{(m+1)}_{n-2} = x^{(m)}_{n-1} x^{(m)}_{n-2} \in \{10, 01\}$, and $\rho_i(x^{(m+1)}, y) = \rho_i(x^{(m)}, y) = 1$ with $x^{(m+1)}_{2i+1} x^{(m+1)}_{2i} = x^{(m)}_{2i+1} x^{(m)}_{2i} = 00$ for $i \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - 1\} - \{0, \frac{j_1}{2}, \frac{j_2}{2}, \frac{j_{k'-1}}{2}, \frac{j_{k'+1}}{2}, \frac{j_{k'+2}}{2}, \ldots, \frac{j_{m+1}}{2}\}$.

According to the above discussion, we have $x^{(\lfloor \frac{n-1}{2} \rfloor)} \in \{01(01)^{\lfloor \frac{n-1}{2} \rfloor + 1}, 10(01)^{\lfloor \frac{n-1}{2} \rfloor + 1}\}$ and, thus, $(x^{(\lfloor \frac{n-1}{2} \rfloor)}, y) \in E(CQ_n)$. Noticing that $\lfloor \frac{n-1}{2} \rfloor = \lceil \frac{n+1}{2} \rceil - 2$, we have $(x^{(\lceil \frac{n+1}{2} \rceil - 2)}, y) \in E(CQ_n)$. Further, since $(x^{(\lceil \frac{n+1}{2} \rceil - 2)}, x^{(\lceil \frac{n+1}{2} \rceil - 1)}) \in P$ and $(x^{(\lceil \frac{n+1}{2} \rceil - 1)}, y) \in P$, we get a cycle y, $x^{(\lceil \frac{n+1}{2} \rceil - 2)}$, $x^{(\lceil \frac{n+1}{2} \rceil - 1)}$, y in CQ_n, whose is length 3, a contradiction.

Case 2. $\lceil \frac{n+1}{2} \rceil$ is odd. Select $x = x_{n-1} x_{n-2} \ldots x_1 x_0 = 0^{n-1} 1$, $y = y_{n-1} y_{n-2} \ldots y_1 y_0 = 1^{n-2} 01$. Similar to Case 1, we can prove that there is no a path of length $\lceil \frac{n+1}{2} \rceil$ between x and y in CQ_n when $n \ge 5$. Thus, the proof is omitted. □

This theorem shows that $\lceil \frac{n+1}{2} \rceil + 1$ is the shortest possible length that can be embedded between arbitrary two distinct nodes with a dilation of 1 in the n-dimensional crossed cube, which implies that this range of path lengths is optimal.

5 Conclusions

In this paper, we have studied embedding of paths of different lengths between any two nodes in crossed cubes. We have proven that paths of all lengths between $\lceil \frac{n+1}{2} \rceil + 1$ and $2^n - 1$ can be embedded between any two distinct nodes with a dilation of 1 in the n-dimensional crossed cubes. We have also proven that $\lceil \frac{n+1}{2} \rceil + 1$ is the shortest possible length that can be embedded between arbitrary two distinct nodes with a dilation of 1 in the n-dimensional crossed cube. The embedding of paths is optimal in the sense that its dilation is 1.

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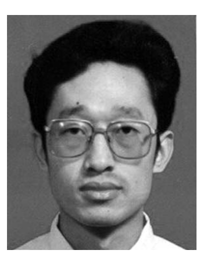

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