FAIR AND EFFICIENT ONLINE ALLOCATIONS:

PART II

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Can we escape?
OLD SETUP

• $n$ additive agents and $T$ indivisible items.
• For each agent $i$ and each item $t \in T$, let $\nu_{it}$ be the preference/valuation agent $i$ for item $t$.
• The items arrive online (one per round) and the agents’ values are revealed when the items arrive (and are chosen by a non-adaptive adversary)
NEW SETUP

- **Two** agents and $T$ divisible items.
- For each agent $i$ and each item $t \in T$, let $v_{it}$ be the preference/valuation agent $i$ for item $t$.
- The items arrive online (one per round) and the agents’ values are revealed when the items arrive (and are chosen by a non-adaptive adversary).
- Agents’ valuations are **normalized** so that $\sum_{t \in T} v_{it} = 1$.
- A fractional allocation $x$ defines fraction $x_{it}$ of item $t$ agent $i$ is going to receive.
- The values are additive, given an allocation rule $x$, the **utility** of agent $i$ is defined as:

$$u_i(x) = \sum_{t \in T} v_{it} x_{it}$$
SETUP

• Our Goal: **Maximize Social Welfare**

\[
\max_x \sum_{i \in N} u_i(x)
\]

• Fairness Constraint: **Fair share**

\[ u_i(x) \geq 1/2 \text{ for all } i \]

• Feasibility Constraint:

\[
\sum_{i \in N} x_{it} \leq 1 \text{ for all } t
\]

• Performance Measurement: we say some algorithm \( \mathcal{A} \) is an \( \alpha \)-approximation to the **optimal social welfare** if:

\[
\min_v \frac{SW\left(x^{\mathcal{A}}(v)\right)}{SW\left(x^{\text{OPT}}(v)\right)} \geq \alpha
\]
# ONLINE VS OFFLINE

<table>
<thead>
<tr>
<th>Allocations</th>
<th>Agent 1</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item 1</td>
<td>1</td>
<td>0</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Values</th>
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<th>Agent 2</th>
</tr>
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<tbody>
<tr>
<td>Item 1</td>
<td>0.9</td>
<td>0.6</td>
</tr>
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Allocations

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<tbody>
<tr>
<td>Item 2</td>
<td>0.1</td>
<td>0.4</td>
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**Fair share is violated**

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<tr>
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<td>0.5</td>
<td>0.5</td>
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Optimal Social Welfare = 0.9 + 0.4 = 1.3

Algorithm Output = 1

\[ \alpha = 76.9\% \]
### ONLINE VS OFFLINE

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<tbody>
<tr>
<td>Item 1</td>
<td>0.9</td>
<td>0.6</td>
<td></td>
<td>5/6</td>
<td>1/6</td>
</tr>
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<td>0.1</td>
<td>0.4</td>
<td></td>
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**Optimal Social Welfare** = 0.9 + 0.4 = 1.3

**Fair-share Optimal Welfare** = 0.75 + 0.5 = 1.25

\[ \alpha = 96.2\% \]
POLY-PROPORTIONAL ALGORITHM

Definition: Poly-proportional algorithms are a family of non-adaptive, anonymous algorithms that allocate an item “proportionally” with some power $p$: $x_{it} = \frac{v_{it}^p}{\sum_{j=1}^{n} v_{jt}^p}$.

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Line of $p$ values:
- $p = 0$: Uniform Fair-share ✓ $\alpha = 0.5$
- $p = 1$: Proportional Fair-share ✓ $\alpha = 0.828$
- $p = \infty$: Max SW Fair-share ❌ $\alpha = 1$
\( p = 1: \text{FAIR SHARE PROOF} \)

- Milne’s inequality:
\[
\sum_{j=1}^{m} \frac{x_j y_j}{x_j + y_j} \leq \frac{(\sum_{j=1}^{m} x_j)(\sum_{j=1}^{m} y_j)}{\sum_{j=1}^{m} x_j + \sum_{j=1}^{m} y_j}
\]

- Plugging in \( x_j = v_{1,j} \) and \( y_j = v_{2,j} \):
- LHS is the value of agent 1 for agent 2’s allocation
- RHS = 1/2
POLY-PROPORTIONAL ALGORITHM

Definition: Poly-proportional algorithms are a family of non-adaptive, anonymous algorithms that allocate an item “proportionally” with some power p: \( x_{it} = \frac{v_{it}^p}{\sum_{j=1}^{n} v_{jt}^p} \)

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Line of p values

- \( p = 0 \) Uniform Fair-share ✓ \( \alpha = 0.5 \)
- \( p = 1 \) Proportional Fair-share ✓ \( \alpha = 0.828 \)
- \( p = 2 \) Fair-share ✓ \( \alpha = 0.894 \)
- \( p > 2 \) Fair-share ✗
- \( p = \infty \) Max SW Fair-share ✗ \( \alpha = 1 \)
Let’s consider $p = 3$ on the following instance:

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<tbody>
<tr>
<td>Item 1</td>
<td>0.6386</td>
<td>0.5</td>
</tr>
<tr>
<td>Item 2</td>
<td>0.3065</td>
<td>0.24</td>
</tr>
<tr>
<td>Item 3</td>
<td>0.0549</td>
<td>0.26</td>
</tr>
</tbody>
</table>

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<th>Allocations</th>
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</tr>
</thead>
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<tr>
<td></td>
<td>0.6757</td>
<td>0.3243</td>
</tr>
<tr>
<td></td>
<td>0.6756</td>
<td>0.3244</td>
</tr>
<tr>
<td></td>
<td>0</td>
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**GUARDED POLY-proportional**

- **Critical Point:** At the end of some round $c$, the utility that agent $i$ has received so far plus her value for all the remaining items is exactly $1/2$, i.e.,

$$\sum_{t=1}^{c} v_{it} x_{it} + \sum_{t=c+1}^{T} v_{it} = 1/2$$

- **Guarded Poly-Proportional Algorithms:** Perform poly-proportional until one of the agent reaches critical point (if there is one), then fully allocates all the remaining items to that agent.

- Lemma: The guarded poly-proportional algorithm with any $p \geq 0$ satisfies fair-share.

- For $p \leq 2$, critical point never occurs

Guarded poly-proportional with $p = 2.7$:

$$\alpha = 0.916$$
**Theorem [GPT 2021]:** There is no online fair-share algorithm that achieves an approximation to the optimal welfare better than 0.933
MULTIPLE AGENTS CASE

• Caragiannis et al. (2012) prove that even if we knew all the values in advance, the *price of fairness* is $O \left( \frac{1}{\sqrt{n}} \right)$.

• The proportional algorithm matches this bound in an online manner, and therefore achieves the optimal approximation.

• If we were to restrict the benchmark to be the *optimal social welfare subject to the fair-share constraint*, still no online algorithm could achieve an approximation better than $\Omega \left( \frac{1}{\sqrt{n}} \right)$. 
Non-adaptive adversary

Can we escape?

Adaptive adversary
Can we get even more?

- Major issue with model so far: **agents must be expressive**
  - Reporting an **exact numerical value** for each item is too much for many applications of interest
OLD SETUP

• $n$ additive agents and $T$ indivisible items.
• For each agent $i$ and each item $t \in T$, let $v_{it}$ be the preference/valuation agent $i$ for item $t$.
• The items arrive online (one per round) and the agents’ values are revealed when the items arrive.
• There is a known distribution $D_i$ for each agent $i$ from which her values are drawn from.
NEW SETUP: PARTIAL INFORMATION
[BHP 2022; UNPUBLISHED]

• $n$ additive agents and $T$ indivisible items.
• For each agent $i$ and each item $t \in T$, let $v_{it}$ be the preference/valuation agent $i$ for item $t$.
• The items arrive online (one per round) and the agents’ values are realized when the items arrive.
• There is a unknown distribution $D_i$ for each agent $i$ from which her values are drawn from.
• Our algorithms never learn the value of an item.
• Instead, we learn the relative rank of agent $i$ for item $t$, with respect to previously allocated items.
What the algorithm knows

Nothing

1

Nothing

1
What the algorithm knows

Nothing

0.5

0.5

>
What the algorithm knows

0.9

0
WHAT CAN WE DO?

• Empirical quantiles will be important for us
• Given a fresh item $t$, we will try to estimate its true quantile value $q_{i,t} = \Pr[ D_i \leq v_{i,t} ]$
• We will do almost as well as an ideal algorithm that has access to true quantiles
IDEAL ALGORITHMS

• **Quantile maximization**: allocate each item to the agent with the highest quantile)

• **q-threshold**: allocate each item uniformly at random among agents whose quantile is at least q.

• **Lemma [DGKPS 14]**: Both algorithms are strongly envy-free with high probability.

• **Lemma**: The \( \frac{n-1}{n} \)-threshold algorithm guarantees a \( \left(1 - \frac{1}{e}\right)^2 \approx 0.4 \) approximation to welfare in the i.i.d. setting

• **Property \( \mathcal{P}^* \)**: if there is exactly one agent whose quantile is at least \( 1 - 1/n \), she gets the item

• **Lemma**: Every algorithm that satisfies \( \mathcal{P}^* \) guarantees a \( 1/e \approx 0.36 \) approximation to welfare in the non-i.i.d. setting.
WHAT CAN WE NOT DO?

- **Theorem**: Even for \( n = 2 \) agents, there is no algorithm \( \mathcal{A} \) that is **one-swap Pareto** efficient and envy-free whp, even when agents values are drawn i.i.d. from a **known** distribution \( D \).

- **Proof sketch**:
  - The first item must be allocated arbitrarily (e.g. to agent 1 wlog), since no information is available.
  - With a constant probability, agent 2 really likes the item (her value is in the top 0.1 quantile), but agent 1 does not (her value is in the bottom 0.1 quantile).
  - Our first decision is an irrevocable mistake: the first item we give to agent 2 has a constant probability of having the opposite quantiles (high for 1, low for 2).
  - And, agent 2 should get items in order to satisfy EF whp.
MATCHING THE IDEAL ALGORITHMS

**Algorithm 1**

- For epoch $k = 1, 2, ...$
  - Explore ($n \cdot k^4$ items):
    - Give $k^4$ to each agent
  - Exploit ($k^8$ items):
    - Each item $g$ goes to the agent with the highest empirical quantile, with respect to the exploration phase of epoch $k$
MATCHING THE IDEAL ALGORITHMS

• Giving \( m \) (random) items to each agent, we can get (probabilistic) bounds on the empirical quantile of fresh items
  ◦ “The sample is \( \epsilon \)-accurate with probability at least \( 1 - \delta \)”
  ◦ \( \epsilon \)-accurate: the relative rank of a fresh item is correct with probability at least \( 1 - \epsilon \)

• However, we still need epochs!

• The underlying distribution is unknown, so we cannot fix a target accuracy even when shooting for a constant approximation to efficiency
MATCHING THE IDEAL ALGORITHMS

Algorithm 1

- For epoch $k = 1, 2, \ldots$:
  - Explore ($n \cdot k^4$ items):
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Lemma: The allocation of Algorithm 1 differs from that of the quantile maximization algorithm after $T$ steps by at most $f(T)$ items, whp, where $f(T) \in O\left(poly(n) \cdot T^{\frac{15}{16}}\right)$. 
Algorithm 1

- For epoch $k = 1, 2, \ldots$:
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**Theorem:** In the i.i.d. model Algorithm 1 gives a $(1 - \epsilon)$-approximation to welfare for all $\epsilon > 0$, and is envy-free, with high probability.
MATCHING THE IDEAL ALGORITHMS

Algorithm 1
• For epoch $k = 1, 2, \ldots$ :
  ◦ Explore $(n \cdot k^4$ items):
    • Give $k^4$ to each agent
  ◦ Exploit $(k^8$ items):
    • Each item $g$ goes to the agent with the highest empirical quantile, with respect to the exploration phase of epoch $k$

Theorem: In the non i.i.d. model Algorithm 1 gives a $1/e$-approximation to welfare for all $\epsilon > 0$, and is envy-free, with high probability.
A MATCHING LOWER BOUND

- Theorem: In the non i.i.d. model, no algorithm is EF and 0.81-PO with probability $p > 2/3$, even for $n = 2$ agents.
- Sketch:
  - We consider two distributions $D_{flat} = U[1 - w, 1]$ and $D_{skewed} = 1$ w.p. $z$ (and 0 w.p. $1 - z$).
  - The algorithm must be EF + $c$-PO with probability $2/3$ at time $t \geq T^*$, for some $T^*$, for each combo of distributions for the agents.
  - With constant probability, at time $t$, the number of items with high quantiles for each agent is near its expectation.
    - E.g. Number of items of agent 1 with quantile at least $1 - z$ is $z \pm \delta$.
  - Via union bound, there must exist a sequence of items, where a number of things happen: the algorithm satisfies the properties for all combos of distributions, and the sample is “nice”.
  - Envy-freeness implies a certain distribution of the high quantile items; we give a Pareto improvement.
Partial information is great

However, perhaps it is still unreasonable to expect comparisons with all previously allocated items

What can we do with a fixed memory?

- An agent can compare fresh items only with items in her memory
- An algorithm in this model can decide to replace an item in memory with a fresh item

What can we do with a memory of one item?
A LOWER BOUND

• **Theorem:** In the i.i.d. model, given a memory of one item per agent, there is no algorithm $\mathcal{A}$ that is 0.999-welfare maximizing with high probability

• So, constant approximations are necessary
Theorem: There exists an algorithm that achieves envy-freeness and a $\left(1 - \frac{1}{e}\right)^2 \approx 0.4$ approximation to welfare, with high probability, in the i.i.d. model.

- The approximation is $1/e$ in the non-i.i.d. model

Algorithm 2 is similar to Algorithm 1: exploration & exploitation phases

- We update the memory, and then check if the quantile of the item in memory is useful (close to the ideal $q^* = 1 - 1/n$)
- But, we need to account for more things going wrong
- Could be that the item in memory is bad, or that the sample we use to check is bad
Demand for more by weakening the algorithms

Escape by weakening the adversary
REFERENCES

• Fair and Efficient Online Allocations with Normalized Valuations. Gkatzelis, Psomas, Tan. AAAI 2021


• The efficiency of fair division. Caragiannis, Kaklamanis, Kanellopoulos, Kyropoulou. TCS 2012
THANK YOU!