FAIR AND EFFICIENT ONLINE ALLOCATIONS:
PART I

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FAIR DIVISION

TEXTBOOK TREATMENT

• INPUT:
  ◦ The resources we are dividing
    • E.g. \( m \) indivisible items
  ◦ The agents and their utility structure
    • E.g. \( n \) additive agents and a value \( v_{i,j} \) for each agent \( i \) and item \( j \)
  ◦ Constraints on the output (fairness, efficiency, etc)
    • E.g. EF1

• OUTPUT:
  ◦ An allocation of the resources that (approximately) satisfies the constraints
FAIR DIVISION

• Standard real-world motivations:
  ◦ Inheritance, Divorce settlements
  ◦ Housing
  ◦ Dividing land/airspace
  ◦ Computational resources
  ◦ Food donations
  ◦ Kidney exchanges
  ◦ Organ/blood donations
FAIR DIVISION

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Not really one-shot problems
## DYNAMIC FAIR DIVISION

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| Hybrids                          | Kidney exchanges                 |
|                                  | Organ/blood donations            |
# DYNAMIC FAIR DIVISION

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This talk

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A FIRST PROBLEM

• There are $n$ additive agents
• Indivisible items arrive over time
  ◦ One in each stage for $T$ stages
• Agent $i$ has value $v_{it} \in [0,1]$ for item $t$ that we learn when the item arrives
My value for my stuff

My value for your stuff
\[ ENVY_{RB}^R = 1.5 - 1 = 0.5 \]

\[ ENVY_{BR}^R = 1 - 0.5 = 0.5 \]
A FIRST PROBLEM

• For the static version, we can keep the maximum envy at most 1 (since $v_{i,t} \in [0,1]$)

• First goal:
  ◦ Minimize the maximum envy at the final step $T$
A MODELING DECISION

• How is $v_{it}$ generated?
  ◦ Classic online algorithms: adaptive and non-adaptive adversary
  ◦ Bayesian adversaries: values are drawn from a distribution
ADVERSARY MODELS

Stronger

\[ v_{it} \sim D \quad v_{it} \sim D_i \quad \tilde{v}_t \sim D \quad \text{Non-adaptive adversary} \quad \text{Adaptive adversary} \]
We write down an algorithm
The adversary decides the items’ values after seeing our code, and the random outcomes of any coin flipping the algorithm does
ADVERSARY MODELS

- We write down an algorithm
- The adversary decides the items’ values after seeing our code, but **not** the random outcomes of the coin flipping the algorithm does
• Items’ values are drawn independently and identically from a known distribution $D$, the same for all agents and all items
ADVERSARY MODELS

- Agent $i$’s values are drawn independently and identically from a known, agent specific distribution $D_i$
At each time step $t$, a vector of values $\tilde{v}_t = (v_{1,t}, ..., v_{n,t})$ is drawn from a known distribution $D$.

Values can be correlated in a given step (but independent over different time steps).
WHAT TO EXPECT: FAIRNESS

• Linear ($\Theta(T)$) envy is trivial
  ◦ E.g. giving all items to the same agent

• Vanishing envy: $\lim_{T \to \infty} \frac{\mathbb{E}[\max_{i,j} ENVY_{T}^{ij}]}{T} = 0$
  ◦ $\mathbb{E} \left[ \max_{i,j} ENVY_{T}^{ij} \right] \in o(T)$

• Envy free up to one item (EF1) with probability 1

• Envy free with high probability
WHAT TO EXPECT: EFFICIENCY

• Pareto efficiency:
  ◦ An allocation is Pareto efficient if there is no allocation where all agents get more utility (with at least one agent getting strictly more utility)

• $\alpha$-Pareto efficiency (Ruhe and Fruhwirth, 1990):
  ◦ An allocation is $\alpha$-Pareto efficient if no allocation improves the utility of all agents by a factor of $1/\alpha$
    • E.g., a dictatorship is $\frac{1}{n} + \epsilon$ Pareto efficient
**Algorithm**: Random allocation

**Fairness**: $\mathbb{E} \left[ \max_{i,j} Envy_{i,j}^T \right] \in \tilde{O}(\sqrt{T/n})$ [BKPP 2018]

**Efficiency**: $\frac{1}{n}$ Pareto efficient ex-ante
Theorem[BKPP 18]: An adaptive adversary can always ensure $\max_{i,j} \text{ENVY}_{T}^{i,j} \in \Omega\left(\sqrt{T/n}\right)$.

- Thus, random allocation is **asymptotically optimal**!
- **Good news:** We can get the same guarantee with a deterministic algorithm!
  - Define a potential function $\phi(t)$
  - Allocate in a way that $\phi(t)$ is minimized
- **Question:** Can we improve the efficiency guarantee while maintaining optimal fairness?
What about efficiency?

Theorem [ZP 20]: There is no algorithm that guarantees vanishing envy \( T^{1-\epsilon} \) and is \( \left( \frac{1}{n} + \epsilon \right) \)-Pareto efficient for any \( \epsilon > 0 \)
PROOF SKETCH OF THEOREM FOR ADAPTIVE ADVERSARY

• Assume algorithm $A$ had envy $f(T) \in o(T)$ on all inputs, and was $\frac{1}{n} + \epsilon$ Pareto efficient.

• **Instance $I^*$:** each agent $i$ has values
  - $v_{i,t} = 1$ for the $T/n$-th segment
    - Items $t \in \left[ \frac{T}{n}(i-1) + 1, ..., \frac{T}{n}i \right]$
  - $v_{i,t} = \epsilon$ for all other items

• An adaptive adversary can always stop showing $I^*$ and make all remaining items worthless

• Therefore envy at step $t$ must be at most $f(T)$ for all agents

• This implies that in each segment $i$, every agent must get $\frac{T}{n^2} \pm \frac{f(T)}{\epsilon} \left( 1 + \frac{2}{\epsilon} \right)^{i-1}$ items

• Thus, final utility for each agent is at most $\left( \frac{T}{n^2} + \frac{f(T)}{\epsilon} \left( 1 + \frac{2}{\epsilon} \right)^{n-1} \right) \cdot (1 + (n-1)\epsilon)$

• But, it is possible to give all agents utility $T/n$
PROOF SKETCH OF THEOREM FOR NON-ADAPTIVE ADVERSARY

• The non-adversary has $n$ instances in their arsenal.

• $I_i$’s first $\frac{T}{n}$ items follow $I^*$, and the rest have zero value.

• Again, we bound the number of items the algorithm can allocate to each agent in each segment.

• The new bound is looser and probabilistic, but gets the job done.
WHAT ABOUT EFFICIENCY?

$v_{it} \sim D$

$v_{it} \sim D_i$

$\hat{v}_t \sim D$

Non-adaptive adversary

Adaptive adversary
INDEPENDENT AND IDENTICAL DISTRIBUTION

Algorithm: Give each item to the agent with the highest valuation

 Guarantees (under mild conditions on $D$) [KPW, AAAI 16]:
• Pareto efficient (ex-post)
• Envy-freeness with high probability
GREEDY ALGORITHM

$U[0,1]$
GREEDY ALGORITHM

- Everyone roughly receives the same number of items
  - But when $i$ receives an item, it is more valuable
  - Chernoff Bounds
- Can replace $U[0,1]$ with any distribution with constant variance
Proposed Algorithm:
Give each item to the agent with the highest quantile??

- $U(0,1)$ and $U(0.49, 0.51)$
  - Agent 2 essentially only cares about the number of items
- This algorithm is envy-free whp, but not efficient
Algorithm [Bai, Gölz 2022]:

- Find $\beta_i$ for each agent $i$, such that $\Pr[\beta_i v_{i,t} = \arg\max_j \beta_j v_{j,t}] = 1/n$
  - i.e. Allocating to $\arg\max_j \beta_j v_{j,t}$ gives $i$ the next item with probability $1/n$

Properties:
- Envy free with high probability
- Pareto optimal (since it maximizes weighted welfare)
Theorem [ZP 20]: There is an ex-post Pareto optimal algorithm that guarantees to each pair of agents $i, j$:
- Either $i$ does not envy $j$ with high probability
- Or, $i$ envies $j$ by at most one item (with probability 1)
Theorem [ZP 20]: There is an ex-post Pareto optimal algorithm that guarantees to each pair of agents $i, j$:

- Either $i$ does not envy $j$ with high probability
- Or, $i$ envies $j$ by at most one item (with probability 1)

Main structural result:
Given $n$ agents and $m$ items, there is a Pareto efficient fractional allocation such that each agent $i$:

- Either strictly prefers her own bundle to the bundle of agent $j$
- Or $i$ and $j$ have identical allocations and the same value for all the items that are allocated to them
INTUITION

• How could you ever be ex-post Pareto?
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w.p. 0.1   w.p. 0.9
INTUITION

• How could you ever be ex-post Pareto?
• Idea 1: every time item 1 comes, give it to red agent, o.w. give item to blue agent
  ◦ Efficient, but not fair!

```
+---+---+
| 2 |   |
+---+---+
|   | 1 |
+---+---+
| 1 | 2 |
+---+---+
```

w.p. 0.1    w.p. 0.9
INTUITION

- How could you ever be ex-post Pareto?
- Issue: if you **ever** allocate item 2 to the red agent, you **cannot** allocate item 1 to the blue agent

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w.p. 0.1
INTUITION

• How could you ever be ex-post Pareto?
• Insight: we should Pareto efficient and fair in the instance where values are multiplied by probabilities

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BLUEPRINT

• Construct this static instance $I$ from the correlated distribution

• Find a fractional allocation $x$

• For the online problem, every time item $k$ comes, allocate to agent $i$ with probability $x_{ik}$
ALGORITHM

• Fact 1: Being Pareto efficient in $I$ turns out to be enough for Pareto efficiency ex-post for the online problem!

• Fact 2: Being envy-free in $I$ will give vanishing envy

• Question: Can we do better?
ALGORITHM

• The dream: Pareto efficiency and strong envy-freeness for $I$
  ◦ Then, Chernoff would give EF w.h.p.
• Pretty much impossible
  ◦ Agents could be identical...
• CISEF:
  ◦ Either agent $i$ strictly prefers her own bundle to the bundle of agent $j$
  ◦ Or $i$ and $j$ have identical allocations and the same value (up to a scaling factor) for all the items that are allocated to either of them
• How?
  ◦ Start from CEEI, and try to create strong-envy, without messing up efficiency
TAKE AWAYS

$v_{it} \sim D$

$v_{it} \sim D_i$

$\tilde{v}_t \sim D$

Non-adaptive adversary

Adaptive adversary
TAKE AWAYS

$v_{it} \sim D$
$v_{it} \sim D_i$
$\tilde{v}_t \sim D$

Non-adaptive adversary
Adaptive adversary

Can we escape?
TAKE AWAYS

Can we escape?
See second half of tutorial.
Can we get even more?

$\mathbf{v}_{it} \sim D$  $\mathbf{v}_{it} \sim D_i$  $\hat{\mathbf{v}}_t \sim D$

Non-adaptive adversary  Adaptive adversary
Can we not cheat?
Can we not cheat?

- What if the adversary distribution can depend on $T$?
- Theorem [Bansal et al. 2020]: $O(\log T)$ envy w.h.p for two agents, against the correlated distribution adversary.
WHAT I DIDN’T TALK ABOUT

• Dynamic resources & static agents & incentives!
  ◦ See references at the end of slides for a biased selection of papers
  ◦ Highlights:
    • Infinite horizons, so more tricks available
    • Artificial currencies
• How to Make Envy Vanish Over Time. Benade, Kazachkov, Procaccia, Psomas. EC 2018
• Fairness-Efficiency Tradeoffs in Dynamic Fair Division. Zeng, Psomas. EC 2020
• From monetary to nonmonetary mechanism design via artificial currencies. Gorokh, Banerjee and Iyer. MOR 2021
• Multiagent mechanism design without money. Balseiro, Gurkan, and Sun. OR 2019
• Dynamic mechanisms without money. Guo and Horner. 2009
• Competitive repeated allocation without payments. Guo, Conitzer, and Reeves. WINE 2019