Myerson's Lemma. Implementable $\Rightarrow$ monotone.

monotone $\Rightarrow$ implementable.

Fix $i$, $b \neq i$.

If $(x,P)$ is truthful, we try to find the regime of $P$.

Case 1. True private value $= V_1$

$b_i \text{ is better than } b_2$

$V_i \cdot x_i(V_i, b_i) - P_i(V_i, b_i) > V_i \cdot x_i(V_i, b_i) - P_i(V_i, b_i)$

$\Rightarrow V_1 \cdot x_1 - P_1 > V_1 \cdot x_1 - P_1$

Case 2. True private value $= V_2$

$b_1 \text{ is better than } b_2$

$\Rightarrow V_2 \cdot x_2 - P_2 > V_2 \cdot x_1 - P_1$

$V_i(x_2 - x_1) = P_2 - P_1 \leq V_2(x_2 - x_1)$.

Choose $V_2 = V_1 + \Delta V$. ($\Delta V > 0$)

$\frac{dP(u)}{du} = V \cdot \frac{dx(u)}{du}$.

$P_i(v) = \int_0^u x'(x)(a) \, da \, + \, C$

$P_i(v, b \neq i) = \int_0^u x'(b, b \neq i) \, da \, + \, C$.
Assumption 1. \( x_i(\mathcal{F}, b_{-i}) \) is deterministic w.r.t. \( b_i \).

Assumption 2. \( b_i = 0 \Rightarrow p_i(b_i) = 0 \).

\[
C = 0.
\]

Prove \((X, u)\) is truthful.

Goal. \( u_i(b_i, b_{-i}) \triangleq b_i \cdot \text{true value} \) and \( \mathcal{V}_i \) is better than \"bid \( b_i \)\".

\[
\begin{align*}
\frac{1}{v_i^*} & \geq \frac{1}{v_i^*} - \frac{1}{v_i^*} \\
\frac{p_i(b_i, b_{-i}) - p(b_i, b_{-i})}{p_i(b_i, b_{-i})} &= \int_{b_i}^{v_i} \frac{x_i(b_i, b_{-i}) - x_i(b_i, b_{-i})}{p_i(b_i, b_{-i})} \, db_i \\
&\leq v_i \int_{b_i}^{v_i} \frac{x_i(b_i, b_{-i}) - x_i(b_i, b_{-i})}{p_i(b_i, b_{-i})} \, db_i \\
&\leq v_i (x_i^* - x_i^*)
\end{align*}
\]

Myerson's Lemma for sponsored search auction.

Case \( k = 3 \).

\[
p_i(b) = \int_{w_1}^{v_i} \frac{x_i(b_i, b_{-i})}{p_i(b_i, b_{-i})} \, db_i.
\]

Assume, \( p_i = 0 \).

\[
\begin{align*}
V_i(x_i - x_i^*) &\leq \frac{pi}{p_1} - \frac{p_2}{p_1} \in V_2(x_i - x_i^*) \\
&\forall V_2 \in [w_1, w_2].
\end{align*}
\]

\[
\begin{align*}
V_i(x_i - x_i^*) &= \frac{w_1(x_i - x_i^*)}{w_1(x_i - x_i^*)} \\
&= \frac{w_2(x_i - x_i^*)}{w_2(x_i - x_i^*)} \\
&= \frac{w_3(x_i - x_i^*)}{w_3(x_i - x_i^*)}
\end{align*}
\]

\[
\begin{align*}
p_2 - p_i &= \frac{w_1(x_i - x_i^*)}{w_1(x_i - x_i^*)} \\
p_2 - p_3 &= \frac{w_2(x_i - x_i^*)}{w_2(x_i - x_i^*)} \\
p_3 - p_1 &= \frac{w_3(x_i - x_i^*)}{w_3(x_i - x_i^*)}
\end{align*}
\]

\[
\begin{align*}
p_2 &\leq w_2 \cdot x_i^* \\
p_2 &= w_3 \cdot x_i^* + w_2(x_i^* - x_i^*) \\
\Rightarrow p_1 &= w_3 \cdot x_i^* + w_2(x_i^* - x_i^*) + w_1(x_i^* - x_i^*)
\end{align*}
\]
### VCG mechanism.

Agent $i$'s utility: $u_i(b) = U_i(x(b)) - p_i(b) = \left[ U_i(w^*) + \sum_{j \neq i} b_j(w^*) \right] - \max \limits_{w} \sum_{j \neq i} b_j(w)$

$w^* = \arg \max \limits_w \sum_{j \neq i} b_j(w)$

Social welfare of allocation $w^*$ not depends on $b_i$.

Objective of Agent $i$ is the same as the objective of designer.