Axiomatization of Meaningful Solutions: Connecting Cooperative Game Theory to Social Network Analysis

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Game Theory

Two Basic Paradigms

• **Non-Cooperative Games:**
  • competition between individual players
  
  **Solution Concepts:** Nash equilibrium
  **Applications:** Market exchange economics

• **Cooperative Games:**
  • group of players
  • coalitional games
  
  **external enforcement of cooperative behavior**
  **Applications:** political science, formation of companies, payoffs of coalitions
Data in Cooperative Game Theory

- **Grand Coalition** - Player Set: \([ N ]\)
- **Group Utilities:**
  \[
  \sigma: 2^N \rightarrow \mathbb{R}, \sigma(\emptyset) = 0
  \]
Mathematical Spaces of Data Models

Three-Player Four-Strategy Games

Five-Player Cooperative Games

allocate the payoff among the players in some fair way
Data in Cooperative Game Theory

• Grand Coalition - Player Set: \([ N ]\)
• Group Utilities:

\[
\sigma: 2^N \rightarrow \mathbb{R} , \sigma(\emptyset) = 0
\]

Features:
• Big Model
• Weighted Hypergraphs
A Basic Solution Concept

• for measuring Individual contribution to coalition games
• for fair allocation of global values

when given group utilities:

$$\sigma : 2^N \rightarrow \mathbb{R} , \sigma(\emptyset) = 0$$
Dimensionality Reduction

• Data - group utilities:
  \[ \sigma: 2^N \rightarrow \mathbb{R}, \sigma(\emptyset) = 0, \]
  i.e., \( \mathbb{R}^{2^N} \)

• Solution Concept:
  \[ \phi: \mathbb{R}^{2^N} \rightarrow \mathbb{R} \]
Dimensionality Reduction

• Data - group utilities:
  \[ \sigma: 2^N \rightarrow \mathbb{R}, \sigma(\emptyset) = 0, \]
  i.e., \( \mathbb{R}^{2^N} \)

• Solution Concept:
  \[ \phi: \mathbb{R}^{2^N} \rightarrow \mathbb{R} \]
  \[ \phi \in \mathbb{R}^N \]
Shapley Values

\[ \text{SV}_\sigma (k) = E_\pi \left[ \sigma (S_{\pi,k} + k) - \sigma (S_{\pi,k}) \right] \]

\( S_{\pi,k} \): players placed before \( k \) according to \( \pi \)

Expected Marginal Contribution
Shapley Values

Why is this meaningful?
Axiomatic Properties

- **Efficiency**
  \[ \sum_k SV_\sigma(k) = \sigma([N]) \]

- **Symmetry**
  if \( \forall S, \sigma(S + i) = \sigma(S + j) \), then \( SV_\sigma(i) = SV_\sigma(j) \)

- **Linearity**
  for an group values \( \sigma \) and \( \tau \), \( SV_{\sigma+\tau} = SV_\sigma + SV_\tau \)

- **Null Player**
  if \( \forall S, \sigma(S + k) = \sigma(S) \), then \( SV_\sigma(k) = 0 \)
Shapley’s Axiomatic Characterization

\[ \phi : \mathbb{R}^{2^N} \rightarrow \mathbb{R} \]

- Efficiency
- Symmetry
- Linearity
- Null Player
Applications to Data & Network Analysis

- Data & Network analysis
- Game Theory
- Scalable Algorithms
Graph Theory in the Age of Networks

- Graph Model
  - Nodes: Webpages, Internet routers, or people
  - Edges: links, connections, or friends
Networks are more than their graph representations
Network Data: Rich and Multi-Faceted
Independent Cascade Model

Kempe-Kleinberg-Tardos
**Independent Cascade Model**

Kempe-Kleinberg-Tardos

Social Network: $G = (V, E)$
Influence Process: $D$
Viral Marketing
Domingos and Richardson
Viral Marketing
Domingos and Richardson
Viral Marketing
Domingos and Richardson
Viral Marketing
Domingos and Richardson
Social Influence
Understanding Multi-Faceted Network Data
What is the impact of an influence process on network centrality?
Network Centrality

- PageRank
- Betweenness
- Local-Sphere of Influence
- ...
Dynamic Processes over Networks
Impact of Influence Dynamics on Network Centrality?

- Influence Process: $D$
- Social Network: $G = (V, E)$

Static measures may not sufficiently capture social-influence centrality
Two-Node Network Influence
The Underlying Interplay

- Probabilistic View: Powerset Networks

\[ P_{G,D}[S,T] \]
The Underlying Interplay

- Probabilistic View: Powerset Networks  
  \[ P_{G,D} [S,T] \]
- Utility View: The Influence Spread (KKT)  
  \[ \sigma_{G,D} (S) = \sum ( |T| P_{G,D} [S,T] ) \]
Game Theoretical View of Social Influence

Social-Influence Cooperative Games:

$$\sigma_{G,D} (S)$$
Shapley Values

$$SV_{\sigma}(k) = \mathbb{E}_{\pi} \left[ \sigma(S_{\pi,k} + k) - \sigma(S_{\pi,k}) \right]$$

$S_{\pi,k}$: players placed before $k$ according to $\pi$

Expected Marginal Contribution
Shapley’s Axiomatic Characterization

\[ \phi : \mathbb{R}^{2^N} \rightarrow \mathbb{R} \]

- Efficiency
- Symmetry
- Linearity
- Null Player
A Game-Theoretical Approach

the impact of an influence process on network centrality:

• Social-Influence Games:
  \[ \sigma_{G,D}(S) \]

• Shapley Centrality:
  \[ \left[ \text{sv}_{\sigma}(v) \right]_v \]
Dimension-Reduction of Network Data

Probabilistic Model

Utility Model

Centrality

What does the Shapley centrality capture?
Fantastic Research Problem

- Graph Theory (Euler, circa 1736)
- Social Influence (1950s, then circ 2002)
- Cooperative Game Theory (1950s)
Network Science in the Age of Big Data

• Mathematically Meaningful
• Algorithmically Scalable
• Experimentally Validatable
Mathematical Question

What does the Shapley value of the cooperative social-influence game reflect?
Mathematical Question

What does the Shapley value of the cooperative social-influence game reflect?

Axiomatic Characterization

Motivated by:
1. Altman and Tennenholtz: PageRank Axioms
2. Palacios-Huerta and Volij: Intellectual Influence
3. Shapley’s Axioms
Representation Theorem

Soundness:
- Social-influence Shapley centrality satisfies Axioms 1-6

Completeness:
- The solution to Axioms 1-6 is unique
**Influence-Centrality Axioms**

1. **Anonymity**: invariant under permutation
Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
**Influence-Centrality Axioms**

1. Anonymity: invariant under permutation
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**Isolated Nodes**

\[
P_{G,D} [S+u, T+u] = P_{G,D} [S, T]
\]

\[
P_{G,D} [u, u] = P_{G,D} [ , ] = 1
\]

\[
P_{G,D} [S, T+u] = 0
\]
Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Nodes: centrality of isolated is 1
Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Nodes: centrality of isolated is 1

Sink Node

\[ P_{G,D}[S+u, T+u] = P_{G,D}[S, T] + P_{G,D}[S, T+u] \]

\[ P_{G,D}[u, u] = P_{G,D}[, ] = 1 \]
**Influence-Centrality Axioms**

1. **Anonymity**: invariant under permutation
2. **Normalization**: average centrality is 1
3. **Isolated Nodes**: centrality of isolated is 1

**Projection of a Sink Node**

\[ P_{G \setminus u, D} [S, T] := P_{G, D} [S, T] + P_{G, D} [S, T + u] \]
Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Node: centrality of isolated is 1
4. Independence of Sink Nodes: sink-node projection preserves centrality of other sink nodes
Bayesian Social Influence

• Social network: \( G = (V, E) \)
• Influence Model:
  – Processes: \( D[1] \ldots D[r] \)
  – A prior distribution: \( \lambda = (\lambda[1] \ldots \lambda[r]) \),

\[
P_{G,D} [S,T] = \sum \lambda[\theta] \ P_{G,D[\theta]} [S,T]
\]
Axiom 5: Bayesian

- Social network: $G = (V, E)$
- Influence Model:
  - Processes: $D[1] \ldots D[r]$
  - A prior distribution: $\lambda = (\lambda[1] \ldots \lambda[r])$

\[
P_{G,D}[S,T] = \sum \lambda[\theta] \ P_{G,D[\theta]}[S,T]
\]

5. Bayesian: social-influence centrality satisfies the linearity-of-expectation principle
Two-Node Network Influence
Likely Parent-Child Influence Model
Nash Bargaining
How to Improve Centrality in a Family?
Two Tiger Parents?
Critical Set Instances

- $R, \nu$
  1. $Pr[R, R+\nu] = Pr[R+\nu, R+\nu] = 1$
  2. $Pr[S, S] = 1$
Bargaining with Two Tiger Parents
Bargaining with Two Tiger Parents
Axiom 6: Bargaining with Critical Sets

6. Bargaining with Critical Sets: the centrality of $v$ is $r/(r+1)$
**Influence-Centrality Axioms**

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Node: centrality of isolated is 1
4. Independence of Sink Nodes: sink-node projection preserves centrality of other sink nodes
5. Bayesian: social-influence centrality satisfies linearity-of-expectation principle
6. Bargaining with Critical Sets: the centrality of v is \( r/(r+1) \)
The social-influence Shapley centrality is the unique centrality measure that satisfies Axioms 1-6.
Our Proof: Simplicity

• Following Myerson’s proof strategy

• Vector Space: \( \{ P_{G,D} [S,T] \}_{S,T} \) (the probability profile)
• A Full-Rank Basis: the critical set instances and extensions
• Linear Maps: axiom-conforming centrality measures
• Uniqueness: for critical set instances and their extensions
Our Proof: Complexity

- More cares than Myerson’s proof of Shapley’s theorem
- Our axiomatic framework is based on the influence model, rather than on influence spread
- The probabilistic profile has higher dimensionality than the influence-spread profile
The Space of Social Influences

Dimensions:

the number of pairs \((S,T)\) satisfying

1. \(S \subseteq T \subseteq V\), and
2. \(S\) not in \(\{\emptyset, V\}\)
The Space of Social Influences

- **Null Instances:**

- **Basis Instances:**

- **General Instances:**
**Implied Properties**

- **Nondiscrimination (Symmetry) Property:**
  - Nodes with the same marginal influence-spread profile have the same Shapley centrality

- **Independence of Irrelevant Alternatives**
  - Disconnected influence components define their own Shapley centrality
Dimension-Reduction of Network Data

- Probabilistic Model
- Utility Model
- Centrality

- Axiomatic analysis of dimension reduction
- Comparative framework
An Empirically Observed Theorem

Symmetric Independent Cascade Model
  – $G$ undirected
  – $p_{uv} = p_{vu}$

Shapley Symmetry of the Symmetric IC Model:
  The Shapley centrality of each node is 1

- Undirected live graph
- Principle of deferred decision
Shapley Symmetry of the Symmetric IC Model

At first glance: surprising and counterintuitive

- limitation of the Shapley centrality?
  - independent of both network structure and symmetric IC edge probabilities.

- limitation of the symmetric IC model?
  - The “pair-wise symmetry and independence” condition is an extreme assumption (that rarely holds for real-world influence propagation).
Sheds Light on both Network Influence and Game-Theoretical Centrality

The Shapley centrality remarkably reveals this symmetry because:

– instead of measuring individual influence spreads in isolation from other nodes
– captures the expected “irreplaceable power” of each node in group influence
– for the symmetric IC model, the equal Shapley centrality exactly points out that all nodes in the network are replaceable if their are equally positioned in a random order
Two Categories of Axioms

- **Principle Axioms:**
  - Anonymity
  - Bayesian

- **Choice Axioms:**
  - Normalization
  - Isolated Node
  - Independence of Sink Nodes
  - Bargaining with Critical Sets
Two Categories of Axioms

- **Principle Axioms:**
  - the essence of common desirable properties

- **Choice Axioms:**
  - succinctly distill the comparative differences between different formulations.
Deterministic Basis for Stochastic Influence

Critical Sets: Many to One Influence
Richer Influence Models

• Cascading Sequences

• Influencing: Stochastic Cascading Profiles
BFS-Propagation

- Networking Broadcasting
Graph-Theoretical Basis of Influence Models

**Theorem:** BFS Propagation Profiles form a basis of all stochastic cascading profiles.

**Axiomatic Characterization:** Under principle Anonymity and Bayesian axioms, the centrality formulation is uniquely characterized by the centrality formulation of layered graphs.
A Systematic Road Map

Road map for the systematic extension of graph-theoretical distance-based centralities to influence-based centralities.
Network analysis

Game Theory

Scalable Algorithms
Interplay Between Dynamic Processes and Network Structures

Shapley centrality:

• Axiomatically characterized by
  – permutation invariance, scaling invariance, Bayesian linearity
  – three simple boundary cases
• Efficient to approximate
• Extensions:
  – Weighted influence models
    • node has different weights, both algorithm and axiomatization can be extended
  – Axiomatization based on influence spread
**Future Directions**

Broader and deeper understanding of game-theoretical approach to network analysis
- Impact of network dynamics on clusterability
- Community identification
- Bounded rationality

Comprehensive/comparative algorithmic and mathematical framework for network analysis
Interplay Between Dynamic Process and Network Structures

What is the impact on network concepts?
Clusterability and Community Characterization

- Conductance
- Cut-ratio
- Modularity
- PageRank Modularity
Holy Grail of Network Science

To understand the network essence that underlies the observed sparse-and-multifaceted network data
Thank You!