Strategyproof Mechanisms for Multiple Facility Location Games

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Public Good Allocation for Strategic Agents with Linear Preferences

- Agents $N = \{1, \ldots, n\}$ on the real line.
- Agent $i$ wants a facility close to $x_i$, which is private information.
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(Randomized) Mechanism

Mechanism $F$ maps reported ideal locations $y = (y_1, \ldots, y_n)$ to (probability distribution over) set(s) of $k$ facilities.
Public Good Allocation for Strategic Agents with Linear Preferences

- Agents $N = \{1, \ldots, n\}$ on the real line.
- Agent $i$ wants a facility close to $x_i$, which is private information.
- Each agent $i$ reports $y_i$ that may be different from $x_i$.

(Randomized) Mechanism

**Mechanism** $F$ maps reported ideal locations $y = (y_1, \ldots, y_n)$ to (probability distribution over) set(s) of $k$ facilities.
Connection Cost

(Expected) distance of agent i’s **true location** to the **nearest** facility:

\[ \text{cost}[x_i, F(y)] = \text{dist}(x_i, F(y)) = \min_{c \in F(y)} |x_i - c| \]
Preferences and Truthfulness

**Connection Cost**

(Expected) distance of agent i’s true location to the nearest facility:

$$\text{cost}[x_i, F(y)] = \text{dist}(x_i, F(y)) = \min_{c \in F(y)} |x_i - c|$$

**Truthfulness**

For any location profile $x$, agent $i$, and location $y$:

$$\text{cost}[x_i, F(x)] \leq \text{cost}[x_i, F(y, x_{-i})]$$
Candidate Facility Locations:

- **Unrestricted**: Any point (esp. agent locations) can be facility.
- **Restricted**: Facilities selected from \( m \) candidate locations \( C \).

Motivation from multi-winner elections: Chamberlin-Courant.

Social Objective:

\[ F(x) \text{ should optimize (or approximate) a given objective function} \]

Social Cost:

\[ \min \sum_{i=1}^{n} \text{cost}\left[x_i, F(x)\right] \]

Social Welfare:

\[ \max \sum_{i=1}^{n} L - \text{cost}\left[x_i, F(x)\right] \]

Maximum Cost:

\[ \min \max \\{ \text{cost}\left[x_i, F(x)\right]\} \]
Variants and Social Efficiency

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**Social Objective**

$F(x)$ should optimize (or approximate) a given **objective function**.

- **Social Cost**: minimize $\sum_{i=1}^{n} \text{cost}[x_i, F(x)]$
- **Social Welfare**: maximize $\sum_{i=1}^{n} (L - \text{cost}[x_i, F(x)])$
- **Maximum Cost**: minimize $\max\{\text{cost}[x_i, F(x)]\}$
Median Mechanism

- **Median** of \((x_1, \ldots, x_n)\): truthful and optimal, when unrestricted.
Median Mechanism

- **Median** of \((x_1, \ldots, x_n)\): **truthful** and **optimal**, when unrestricted.
- Candidate location closest to \(\text{med}(x_1, \ldots, x_n)\): truthful and **3-approximate**, when restricted.
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- **Median** of \((x_1, \ldots, x_n)\): **truthful** and **optimal**, when unrestricted.
- Candidate location closest to \(\text{med}(x_1, \ldots, x_n)\): truthful and **3-approximate**, when restricted.
  - OPT social cost \(\approx n/4\). OPT social welfare \(\approx 3n/4\).
  - Median social cost \(\approx 3n/4\). Median social welfare \(\approx n/4\).

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Anonymity and truthfulness iff generalized median [Moulin 80]
Median Mechanism

- **Median** of \((x_1, \ldots, x_n)\): **truthful** and **optimal**, when unrestricted.
- Candidate location closest to \(\text{med}(x_1, \ldots, x_n)\): truthful and **3-approximate**, when restricted.
  - OPT social cost \(\approx \frac{n}{4}\). OPT social welfare \(\approx \frac{3n}{4}\).
  - Median social cost \(\approx \frac{3n}{4}\). Median social welfare \(\approx \frac{n}{4}\).
- Anonymity and truthfulness iff **generalized** median [Moulin 80]
Optimal Sensitive to Deviations!

The optimal solution for social cost (and welfare) is **not truthful**!
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Three (+ One) Roads to Truthfulness (with Reasonable Efficiency)

- **Order Statistics**: (generalized) median, two-extremes, percentile mechanisms.
- **Align Incentives** with Optimal (for maximum cost): (randomized) equal-cost mechanism.
- **Restriction to Stable** instances: optimal, almost rightmost, random.
A Tale about Truthfulness in $k$-Facility Location

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- **Align Incentives** with Optimal (for maximum cost): (randomized) equal-cost mechanism.
- **Restriction to Stable** instances: optimal, almost rightmost, random.
- **Winner Imposing** verification: if declared location gets facility, agent must be served by that [F. Tzamos, WINE 10]
Percentile Mechanisms [Sui Boutilier Sandholm, IJCAI 13]

Optimal is not truthful: optimal clustering sensitive to deviations!

Percentile mechanisms are anonymous and truthful (only one?).

For any $k \geq 2$, $(\frac{1}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k - 1}{2^k})$-percentile mechanism is $(1 + O(\frac{1}{k}))$-approximate for social welfare [F. Gourvès Monnot, WINE 16].
Percentile Mechanisms [Sui Boutilier Sandholm, IJCAI 13]

Optimal is \textbf{not} truthful: optimal clustering \textit{sensitive} to deviations!

\((\alpha_1, \ldots, \alpha_k)\)-\textit{percentile} mechanism \((0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq 1)\):

- \(\text{vote}(\ell) = \#\text{agents preferring } \ell \in C \text{ to other candidates in } C\).
- \(j\)-th facility at leftmost \(\ell \in C\) with \(\geq \alpha_j\) \textit{fraction} of vote on \(\ell\) and its left.

  - Median is 0.5-percentile. Two-Extremes is (0, 1)-percentile.

\[n = 80 \quad k = 4 \quad (0.1, 0.3, 0.5, 0.9)\)-percentile\]
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  - Median is 0.5-percentile. Two-Extremes is \((0, 1)\)-percentile.

Percentile mechanisms are anonymous and truthful (only one?).

For any \(k \geq 2\), \((1/(2k), 3/(2k), \ldots, (2k - 1)/(2k))\)-percentile mechanism is \((1 + O(1/k))\)-approximate for social welfare [F. Gourvés Monnot, WINE 16].
Truthful Location of 2 Facilities

Two-Extremes is **truthful** and \((n - 2)\)-**approximate** (best possible).

[Procaccia Tenneholtz, EC 09], [F. Tzamos, ICALP 13]
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Truthful Location of \(k \geq 3\) Facilities

- **Deterministic** anonymous mechanisms have unbounded (in terms of \(n\) and \(k\)) approximation ratio [F. Tzamos, ICALP 13]
- Best known **randomized** mechanism is \(n\)-**approximate** for social cost [F. Tzamos, EC 13]
Equal-Cost Mechanism for $k$-Facility Location

- **Optimal maximum** cost of declared instance $= C/2$.
- **Cover** all agents with $k$ disjoint intervals of length $C$.

Agents' Cost and Approximation Ratio

Agent $i$ has expected cost $= (C - x_i)/2 + x_i/2 = C/2$.

Approx. ratio: 2 for the maximum cost, $n$ for the social cost.
Equal-Cost Mechanism for \( k \)-Facility Location

- **Optimal maximum** cost of declared instance = \( C/2 \).
- **Cover** all agents with \( k \) disjoint intervals of length \( C \).
- Place a facility to an **end** of each interval.
  - With prob. \( 1/2 \), facility at L - R - L - R - . . .
  - With prob. \( 1/2 \), facility at R - L - R - L - . . .
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\[
\begin{align*}
\text{probability 0.5} & \quad \text{probability 0.5} \\
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\end{align*}
\]

\[
\begin{align*}
 x_1 & \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_i \quad \ldots \quad x_{n-1} \quad x_n
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- Agent \( i \) has expected cost = \((C - x_i)/2 + x_i/2 = C/2\).
- Approx. ratio: 2 for the **maximum** cost, \( n \) for the **social** cost.
Agents do not have incentive to lie and increase optimal maximum cost, i.e. $C/2$. Let agent $i$ declare $y_i$ and decrease optimal maximum cost to $C'/2 < C/2$. Then, $i$'s expected cost $= 1/2 C + 1/2 (C - C') > C/2$. 

\[ x_1 \ x_2 \ x_3 \ x_4 \ y_i \ x_i \] 

\[ \text{length } C \] 

\[ \text{length } C' \]
Truthfulness

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- Let agent $i$ declare $y_i$ and decrease optimal maximum cost to $C'/2 < C/2$.
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Two-Extremes is \((n - 2)\)-approximate and best possible.  
[Procaccia Tenneholtz, EC 09], [F. Tzamos, ICALP 13]

Truthful Location of \(k \geq 3\) Facilities

- **Deterministic** anonymous mechanisms have **unbounded** approximation ratio [F. Tzamos, ICALP 13]
- Best known **randomized** mechanism is \(n\)-approximate  
  [F. Tzamos, EC 13]
- Bounded approximation requires facility in **each optimal** cluster. But optimal clustering is **sensitive** to agent deviations.
- Focus on instances with **stable** optimal clustering.
Perturbation Stability in Clustering [Bilu Linial, ITCS 10]

- \( \gamma \)-stability: scaling down any distances by factor \( \leq \gamma \) (while maintaining metric property) does not affect optimal solution.

For \( \gamma \geq 2 \), \( k \)-Facility Location solvable in poly-time.

For \( \gamma \leq 2 - \varepsilon \), \( k \)-Facility Location remains hard.

Real-world instances are supposed to be stable: "Clustering is hard when it doesn't matter" [Roughgarden 17].
Perturbation Stability in Clustering [Bilu Linial, ITCS 10]

- **γ-stability**: scaling down any distances by factor $\leq \gamma$ (while maintaining metric property) does not affect optimal solution.

- For $\gamma \geq 2$, (metric) $k$-Facility Location solvable in poly-time!
  
  [Angelidakis Makarychev Makarychev, STOC 17]

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**Question** [F. Patsilinakos, WINE 21]

Assume that "true" instances are indeed stable. How much stability for **truthfulness** and **reasonable** approximation?
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Assume that “true” instances are indeed stable.
How much stability for *truthfulness* and *reasonable* approximation?

Some Negative Observations

- Optimal solution *not truthful* for any stability $\gamma \geq 1$. 
Question [F. Patsilinakos, WINE 21]
Assume that “true” instances are indeed stable.
How much stability for truthfulness and reasonable approximation?

Some Negative Observations
- Optimal solution not truthful for any stability $\gamma \geq 1$.
- For $k \geq 3$, deterministic anonymous truthful mechanisms for $(\sqrt{2} - \varepsilon)$-stable instances have unbounded approximation (based on [F. Tzamos, ICALP 13])
Remedy and Main Results

- **Optimal** clustering \((C_1, \ldots, C_k)\) due to bounded approximation.
- Stability verification (necessary cond.): allocate facilities only if
  \[ \max\{\text{diam}(C_i), \text{diam}(C_{i+1})\} < d(C_i, C_{i+1}) \]

For \((\sqrt{2} + 3)\)-stable instances without singleton clusters, optimal solution is truthful.

For 5-stable instances, facility at the second from the right in each optimal cluster is truthful and \((n - 2)/2\)-approximate.

For 5-stable instances, facility at a random agent in each optimal cluster is truthful and 2-approximate.
Truthful $k$-Facility Location in Stable Instances

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- For 5-stable instances, facility at **random** agent in each optimal cluster is **truthful** and **2-approximate**.
Optimal Mechanism and Approach to Truthfulness

If optimal clustering \((C_1, \ldots, C_k)\) has **singleton** clusters or 
\[\max\{\text{diam}(C_i), \text{diam}(C_{i+1})\} \geq d(C_i, C_{i+1})\], do **not allocate** facilities!

Otherwise, facilities at \((\text{med}(C_1), \ldots, \text{med}(C_k))\).
Optimal Mechanism and Approach to Truthfulness

If optimal clustering \((C_1, \ldots, C_k)\) has **singleton** clusters or \(\max\{\text{diam}(C_i), \text{diam}(C_{i+1})\} \geq d(C_i, C_{i+1})\), do **not allocate** facilities!

Otherwise, facilities at \((\text{med}(C_1), \ldots, \text{med}(C_k))\).

- Key deviation: rightmost agent of \(C_i\) deviates to \(C_j\), causing \(C_j\) to **split** and \(C_i\) to **merge** with \(C_{i+1}\).
- “Simulate” increase in cost of \(C_j\) by \(\gamma\)-perturbation and decrease in cost of \(C_i\) by agent’s **cost improvement**.
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- “Simulate” increase in cost of \(C_j\) by \(\gamma\)-perturbation and decrease in cost of \(C_i\) by agent’s **cost improvement**.
- Stability: optimal clustering **not affected** by deviation.
Increase Stability to $\gamma \geq 5$ to Resist Singleton Deviations

If optimal $(C_1, \ldots, C_k)$ has $\max\{\text{diam}(C_i), \text{diam}(C_{i+1})\} \geq d(C_i, C_{i+1})$, do not allocate facilities!

Almost Rightmost: Facility at second to the right in each optimal $C_i$.
Random: Facility at random in each optimal $C_i$. 

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Strategyproof Mechanisms for Multiple Facility Location
Increase Stability to $\gamma \geq 5$ to Resist Singleton Deviations

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- Cluster merge not profitable due to robust facility allocation.
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If optimal $(C_1, \ldots, C_k)$ has $\max\{\text{diam}(C_i), \text{diam}(C_{i+1})\} \geq d(C_i, C_{i+1})$, do not allocate facilities!

**Almost Rightmost**: Facility at second to the right in each optimal $C_i$.

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- Cluster merge not profitable due to robust facility allocation.
- 5-stable instances: $x \in C_i$ needs to deviate by $\geq \text{diam}(C_i)$ for singleton cluster.
- $x \in C_i$ cannot deviate to singleton and be served by that facility.
Restriction to Stable Instances Necessary

“Global” Truthfulness and Bounded Approximation Only for Stable

$\gamma$-nice mechanism $\equiv$ deterministic mech. truthful for all instances with bounded approximation (in terms of $n, k$) only for $\gamma$-stable instances.

For any $k \geq 3$ and any $\gamma \geq 1$, there are no anonymous $\gamma$-nice mechanisms for $k$-Facility Location (even on the line).

Well-Separated Instances

Instance with $k + 1$ agents is well-separated if it consists of $k - 1$ isolated and 2 nearby agents.

well-separated instance for $k = 3$
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well-separated instance for $k = 3$
Consistent Allocation for Well-Separated Instances

The Nearby Agents Slide on the Left

- Let $x$ be a well-separated instance with $k$-th facility on $x_{k+1}$.
- For any well-separated instance $x' = (x_{\{k,k+1\}}, x'_k, x'_{k+1})$ with $x'_{k+1} \leq x_{k+1}$, $k$-th facility stays with $x'_{k+1}$.

$k = 3$

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Strategyproof Mechanisms for Multiple Facility Location
Consistent Allocation for Well-Separated Instances

The Nearby Agents Slide on the Left

- Let \( x \) be **well-separated** instance with \( k \)-th facility on \( x_{k+1} \).
- For any well-separated instance \( x' = (x_{-\{k,k+1\}}, x'_k, x'_{k+1}) \) with \( x'_{k+1} \leq x_{k+1} \), \( k \)-th facility stays with \( x'_{k+1} \).

The Nearby Agents Slide on the Right

- Let \( x \) be **well-separated** instance with \( k \)-th facility on \( x_k \).
- For any well-separated instance \( x' = (x_{-\{k,k+1\}}, x'_k, x'_{k+1}) \) with \( x_k \leq x'_k \), \( k \)-th facility stays with \( x'_k \).

\( k = 3 \)

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \]
No Anonymous $\gamma$-Nice Mechanisms for $k \geq 3$

**Theorem**

For any $k \geq 3$ and any $\gamma \geq 1$, there are no anonymous $\gamma$-nice mechanisms for $k$-Facility Location (even on the line).
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Proof Sketch for $k = 3$ and $n = 4$

- **Option set** $I_3(x_{-3}) = \{a : F(x_{-3}, y) = a$ for some location $y\}$
  Set of locations where a facility can be forced by agent 3 in $x_{-3}$.
- $F$ truthful iff all agents get the best in their option set.
Theorem

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![Diagram showing the proof sketch for $k = 3$ and $n = 4$]
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For any $k \geq 3$ and any $\gamma \geq 1$, there are no anonymous $\gamma$-nice mechanisms for $k$-Facility Location (even on the line).

Proof Sketch for $k = 3$ and $n = 4$

- $F$ truthful iff all agents get the best in their option set.
- Contradiction to consistent allocation for well-separated inst.!
Approaches to **truthfulness** with reasonable **efficiency**:
- Order statistics – percentile mechanisms.
- Align incentives with optimal – randomized equal cost mechanism.
- Restriction to stable instances – optimal, almost rightmost, random.
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Close the gap in stability for bounded approximation:
- Lower bound of $\sqrt{2}$ and upper bound of $2 + \sqrt{3}$ (or 5).

Extension to trees and general metrics [F. Pats. Terzoglou 22].

Equal Cost and Random to get randomized truthful for all instances with improved approximation for $\gamma$-stable?
Summary - Open Questions

- Approaches to **truthfulness** with reasonable **efficiency**:
  - Order statistics – percentile mechanisms.
  - Align incentives with optimal – randomized equal cost mechanism.
  - Restriction to stable instances – optimal, almost rightmost, random.

- Close the **gap in stability** for bounded approximation:
  lower bound of $\sqrt{2}$ and upper bound of $2 + \sqrt{3}$ (or 5).

- Extension to **trees** and **general metrics** [F. Patsil. Terzoglou 22].

- Equal Cost and Random to get randomized **truthful** for all instances with improved approximation for $\gamma$-stable?

- Complexity of **determining** whether a $k$-Facility Location instance is $\gamma$-stable, esp. for line and trees?
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instance is $\gamma$-stable, esp. for line and trees?

Learning-augmented truthful mechanisms for $k \geq 3$ facilities.
[Xu Lu, 22], [Agrawal Balkansi Gkatzelis Ou Tan, EC 22] for $k \in \{1, 2\}$. 
Thank You!