Cooperative Game Theory Tutorial

Reshef Meir

Game theory

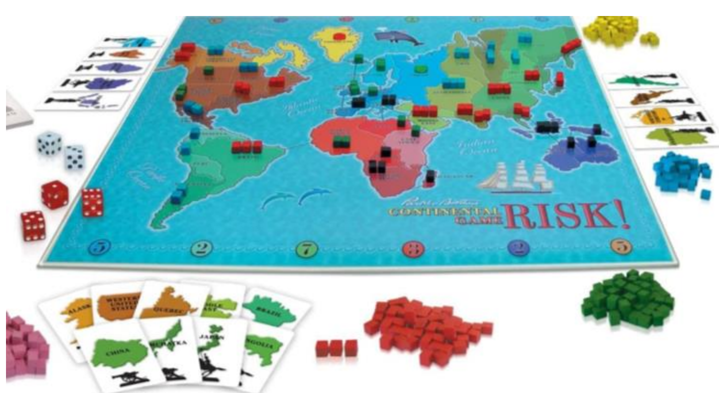
Non-cooperative Game theory

- actions are taken by **individual** agents
- No binding agreements

7 players

Actions: attack, move, ship…

Utilities: Acquired land. Affected by the joint actions of everyone.



Cooperative/coalitional Game theory

- actions are taken by **groups** of agents
- **binding agreements** are possible

7 players

"Actions": form coalitions

Value: Maximum land the coalition can guarantee

Game theory

Non-cooperative Game theory

– actions are taken by **individual** agents
– No binding agreements

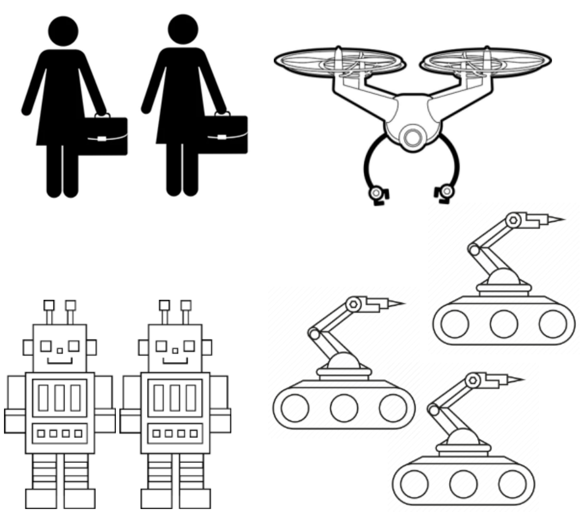
Cooperative/coalitional Game theory

– actions are taken by **groups** of agents
– **binding agreements** are possible

Players: rescue workers and robots of different types

Actions: drill, seek, dig, call, pull, move, …

Utilities: ?



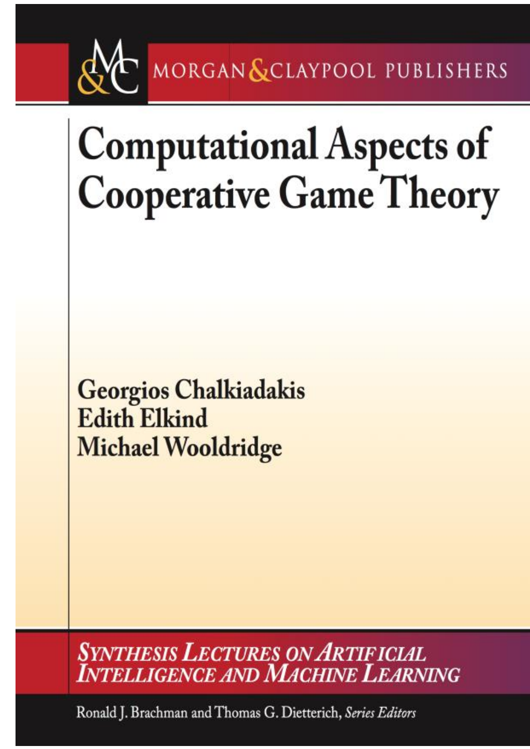
Players: rescue workers and robots of different types

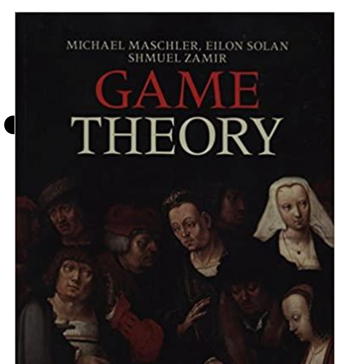
“Actions”: form coalitions

Value: ?

Main Reference

- "Computational Aspects of Cooperative Game Theory"
- Chalkiadakis, Elkind, Wooldridge
- Published by Morgan&Claypool in 2011
- Available online
- This tutorial is based on authors' slides





- advanced material: Machler Zamir Solan

Phases of a Coalitional Game

- Agents form coalitions (teams)
- Each coalition implicitly chooses its action
- Transferable utility (TU) games: the choice of coalitional actions (by all coalitions) determines the payoff of each coalition
  - the members of the coalition then need to divide this joint payoff

Example 1: Buying Ice-Cream

- n children, each has some amount of money
  - the i-th child has b_i dollars
- three types of ice-cream tubs are for sale:
  - Type 1 costs \$7, contains 500g
  - Type 2 costs \$9, contains 750g
  - Type 3 costs \$11, contains 1kg

- children have utility for ice-cream, and do not care about money
- The payoff of each group: the maximum quantity of ice-cream the members of the group can buy by pooling their money
- The ice-cream can be shared arbitrarily within the group

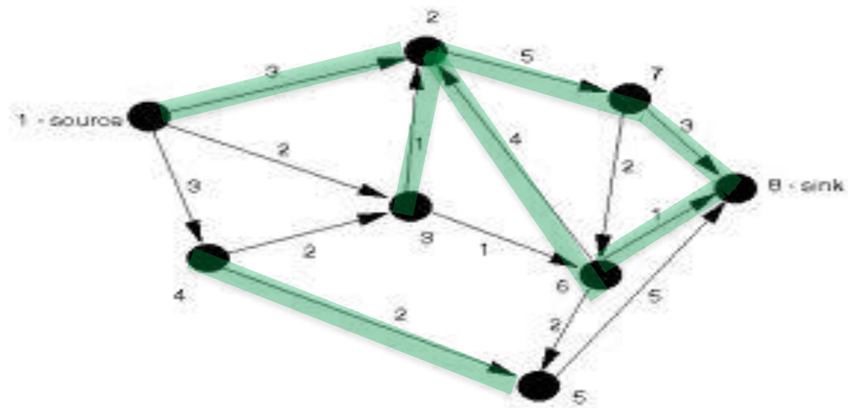
Example 2: Search-and-Rescue by teams of robots

- n robots, each has a set of skills (climb, dig, etc.)
- Each rescue scenario requires a set of skills
- The value of a team of k robots, is the number of different rescue scenarios it can handle

- What is the best partition to teams?
- If robots are made by different companies, how much each company should get?

Example 3: Flow games

- Each agent controls an edge (or several edges) in a weighted flow graph



- <u>Quantity version</u>: The value of a coalition is the amount it can flow from source to target
- <u>Threshold version</u>: The value is 1 if the coalition can flow more than q, and 0 otherwise.

Challenges in TU games

- Representation
  - How to represent the values of all 2^n coalitions?
- Coalition formation
  - What coalitions are likely to form?
- Payoff allocation

Part II:
Stable allocations
the core

Part III:
Fair allocations
Shapley value

Part I: Definitions and Examples

Transferable Utility Games Formalized

- A transferable utility game is a pair (N, v), where:
  - $N = \{1, \ldots, n\}$ is the set of players
  - $v: 2^N \to \mathbb{R}$ is the characteristic function
    - for each subset of players C, $v(C)$ is the amount that the members of C can earn by working together
  - usually it is assumed that v is
    - normalized: $v(\emptyset) = 0$
    - non-negative: $v(C) \geq 0$ for any $C \subseteq N$
    - monotone: $v(C) \leq v(D)$ for any C, D such that $C \subseteq D$
- A coalition is any subset of N;
  N itself is called the grand coalition

Ice-Cream Game: Characteristic Function


C: \$6,

M: \$4,

P: \$3

$w = 500$
$p = \$7$

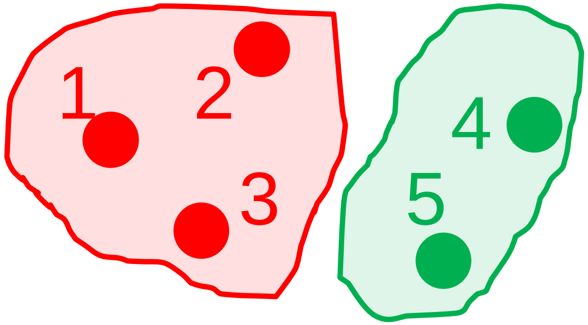
$w = 750$
$p = \$9$

$w = 1000$
$p = \$11$

- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 750$, $v(\{C, P\}) = 750$, $v(\{M, P\}) = 500$
- $v(\{C, M, P\}) = 1000$

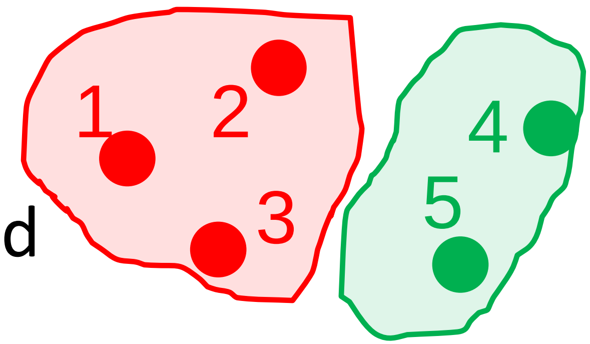
Transferable Utility Games: Outcome

- An outcome of a TU game $G = (N, v)$ is a pair $(CS, \underline{\mathbf{x}})$, where:
  - $CS = (C_1, \ldots, C_k)$ is a coalition structure, i.e., partition of N into coalitions:
    - $\cup_i C_i = N$, $C_i \cap C_j = \emptyset$ for $i \neq j$
  - $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$ is a payoff vector, which distributes the value of each coalition in CS:
    - $x_i \geq 0$ for all $i \in N$
    - $\Sigma_{i \in C} x_i = v(C)$ for each C is CS



Transferable Utility Games: Outcome

- Example:
  - suppose v({1, 2, 3}) = 9, v({4, 5}) = 4
  - then (({1, 2, 3}, {4, 5}), (3, 3, 3, 3, 1)) is an outcome
  - (({1, 2, 3}, {4, 5}), (2, 3, 2, 3, 3)) is NOT an outcome: transfers between coalitions are not allowed



- An outcome (CS, **<u>x</u>**) is called an imputation if it satisfies individual rationality: $x_i \geq v(\{i\})$ for all $i \in N$
- Notation: we will denote $\Sigma_{i \in C}\, x_i$ by $x(C)$

Some classes of games with examples

- Simple games

- Superadditive games

- Convex (supermodular) games

Simple Games

- Definition: a game $G = (N, v)$ is simple if
  - $v(C) \in \{0, 1\}$ for any $C \subseteq N$
  - v is monotone: if $v(C) = 1$ and $C \subseteq D$, then $v(D) = 1$
- A coalition C in a simple game is said to be winning if $v(C) = 1$ and losing if $v(C) = 0$

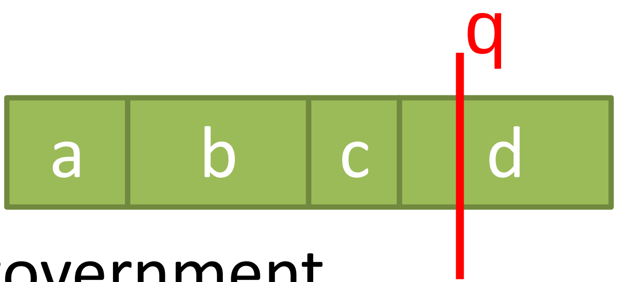
Examples:

- The ice cream game with 1 pack
- Weighted Voting Games

$w = 750$

$p = \$9$

Weighted Voting Games



- n parties in the parliament
- Party i has w_i representatives
- A coalition of parties can form a government only if its total size is at least q
  - usually $q \geq \lfloor \Sigma_{i=1, \ldots, n} w_i /2 \rfloor + 1$: strict majority
- Notation: $w(C) = \Sigma_{i \in C} w_i$
- This setting can be described by a game $G = (N, v)$, where
  - $N = \{1, \ldots, n\}$
  - $v(C) = 1$ if $w(C) \geq q$ and $v(C) = 0$ otherwise
- Observe that weighted voting games are simple games
- Notation: $G = [q; w_1, \ldots, w_n]$
  - q is called the quota

Weighted Voting Games: UK

- United Kingdom, 2005:
  - 650 seats, $q = 326$
  - Conservatives (C): 225
  - Labour (L): 325
  - Liberal Democrats (LD): 62
  - 8 other parties (O), with a total of 38 seats
- $N = \{C, L, LD, O_1, \ldots, O_8\}$
- for any $X \subseteq N$, $v(X) = 1$ if and only if $L \in X$
- L is a veto player

Superadditive and Convex Games

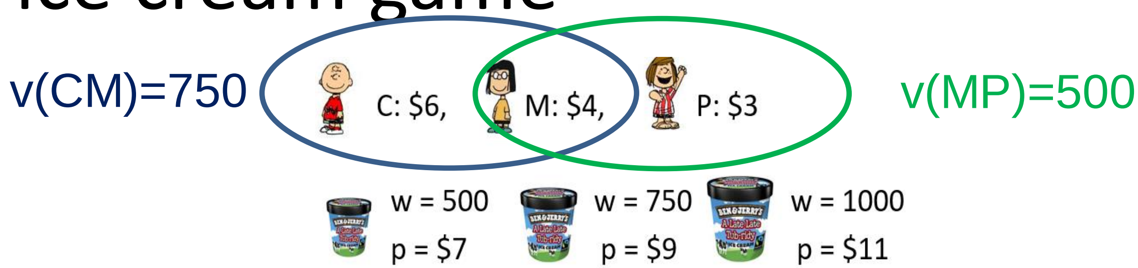
- Definition: a game $G = (N, v)$ is called superadditive if $v(C \cup D) \geq v(C) + v(D)$ for any two disjoint coalitions C and D
- Definition: a game $G = (N, v)$ is called convex if $v(C \cup D)+v(C \cap D) \geq v(C) + v(D)$ for any two coalitions C and D
- In convex [superadditive] games, two [disjoint] coalitions can always merge without losing value; hence, we can assume that players form the grand coalition N

Examples

- Example 1: $v(C) := |C|^2$
  - Convex since

  $v(C \cup D) = (|C|+|D|)^2 \geq |C|^2+|D|^2 - |C \cap D| = v(C) + v(D) - v(C \cap D)$

- Example 2: The ice cream game
  - Not convex



  $v(CM \cup MP) + v(CM \cap MP) = 1000 + 0 < v(CM)+v(MP)$

  - Superadditive since any two disjoint sets can still buy the same amount of ice cream (avoid pooling money)

Part II: Stability

Overview

- The Core
- Examples in different types of games
- Games on graphs

What Is a Good Outcome?

-  C: $4,  M: $3,  P: $3  w = 500 p = $7  w = 750 p = $9
- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{P\}) = 0$
- $v(\{C, M\}) = 500$, $v(\{C, P\}) = 500$, $v(\{M, P\}) = 0$
- $v(\{C, M, P\}) = 750$
- This is a superadditive game
  - Grand coalition is formed (buy the medium pack)
  - outcomes are payoff vectors
- How should the players share the ice-cream?
  - if they share as (200, 200, 350), Charlie and Marcie can get more ice-cream by buying a 500g tub on their own, and splitting it equally
  - the outcome (200, 200, 350) is not stable!

Transferable Utility Games: Stability

- <u>Definition</u>: the core of a game is the set of all stable outcomes, i.e., outcomes that no coalition wants to deviate from

$$\text{core}(G) = \left\{ \underline{\mathbf{x}} \;\middle|\; \begin{array}{ll} x_i \geq 0 \quad \text{for any } i \text{ in } N, & \\ \Sigma_{i \in N}\, x_i = v(N), & \text{(all value is allocated)} \\ \Sigma_{i \in C}\, x_i \geq v(C) \text{ for any } C \subseteq N & \text{(no deviations)} \end{array} \right\}$$

  – each coalition earns at least as much as it can make on its own

Ice-Cream Game: Core

-  C: \$4,  M: \$3,  P: \$3  w = 500, p = \$7  w = 750, p = \$9
- v(Ø) = v({C}) = v({M}) = v({P}) = 0, v({C, M, P}) = 750
- v({C, M}) = 500, v({C, P}) = 500, v({M, P}) = 0
- (200, 200, 350) is not in the core:
  - $v(\{C, M\}) > x_C + x_M$
- (250, 250, 250) is in the core:
  - no subgroup of players can deviate so that each member of the subgroup gets more
- (750, 0, 0) is also in the core:
  - Marcie and Pattie cannot get more on their own!

Games with Empty Core

- The core is a very attractive solution concept
- However, some games have empty cores

 C: \$6,  M: \$5,  P: \$4  $w = 750$

$p = \$9$

  - consider an outcome **x**
  - $x_i > 0$ for some i, so $x(N\setminus\{i\}) < 750$, yet $v(N\setminus\{i\}) = 750$
- There are also other ways to define stable solutions – this class will focus on the core

The Core - Overview

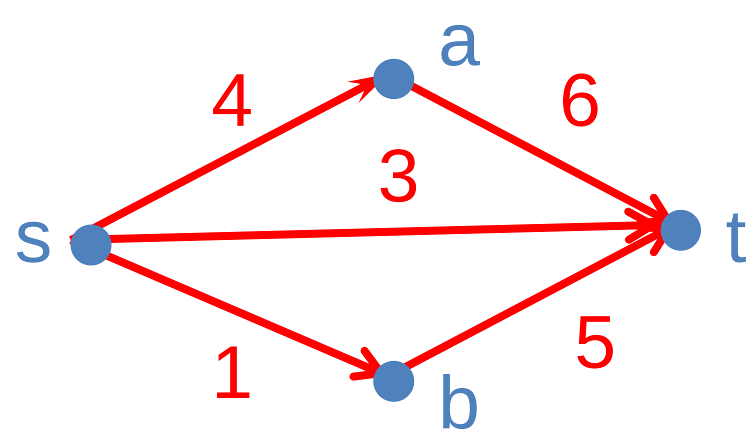
- Definition
- Examples in different types of games
  - Routing (flow) games
  - Weighted voting games
  - Induced subgraph games
  - Assignment games
- General characterization of the core?
- Restricted cooperation

Weighted Voting Games

- WVG are simple games
- Computing the core/checking if an outcome is in the core:
  - Equivalent to check who are the veto players
  - Player i is veto, iff $w(N \setminus \{i\}) < q$
  - Easy to compute

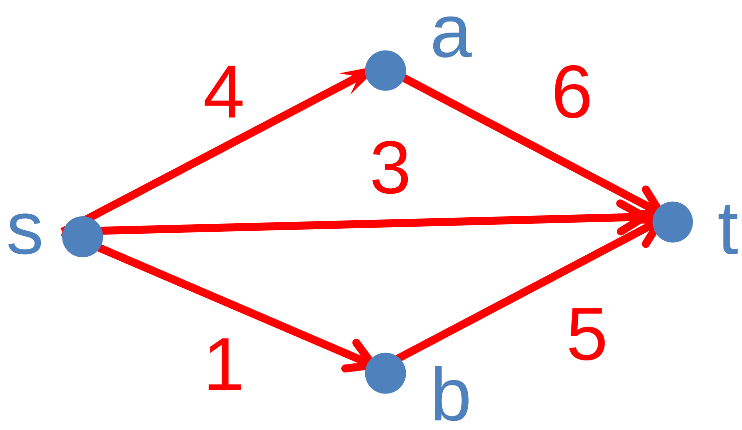
Network Flow Games

- Agents are **edges** in a network with source s and sink t
  - edge e_i has capacity c_i
- Value of a coalition = amount of s–t flow it can carry
  - v({sa, at}) = 4, v({sa, at, st}) = 7
- How to compute the value of a coalition?
- Can the core be empty?
  - Find a min-cut $A \subseteq N$
  - Pay $x_i = c_i$ to each $e_i \in A$
  - <u>**X**</u> is in the core



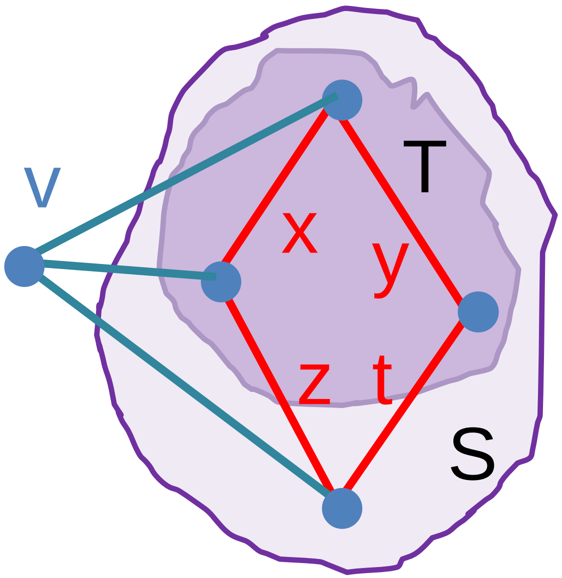
(Threshold) Network Flow Games

- Thresholded network flow games (TNFG): there exists a threshold T such that
  - v(C) = 1 if C can carry ≥ T units of flow
  - v(C) = 0 otherwise
- TNFG with T = 6
  - v({sa, at}) = 0, v({sa, at, st}) = 1
- WVG are just simple TNFG!
  - Parallel edges
  - Core may be empty



Induced Subgraph Games

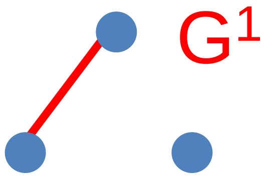
- Players are vertices of a weighted graph
- Value of a coalition = total weight of internal edges
  - $v(T) = x+y$, $v(S) = x+y+z+t$
- Models social networks
  - Facebook, LinkedIn
  - cell phone companies with free in-network calls

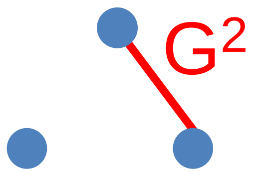


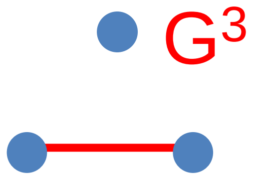
Induced Subgraph Games: Core [Deng, Papadimitriou'94]

- If all edge weights are non-negative, the core is non-empty

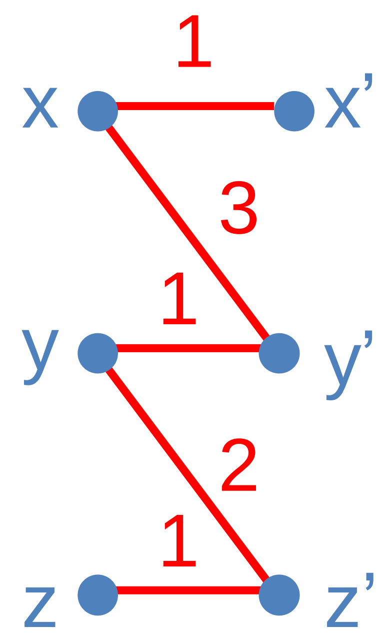
- If weights can be negative, the game is not monotone
  - Theorem: Core is empty iff there is a negative cut







Assignment Games [Shapley & Shubik'72]



- Players are vertices of a bipartite graph (V, W, E)
- Value of a coalition = weight of the max-weight induced matching
  - v({x, y, z}) = 0, v({x, x', y'}) = 3
- Generalization: matching games
  - same definition, but the graph need not be bipartite

General characterizations?

- Simple Games

- Convex games

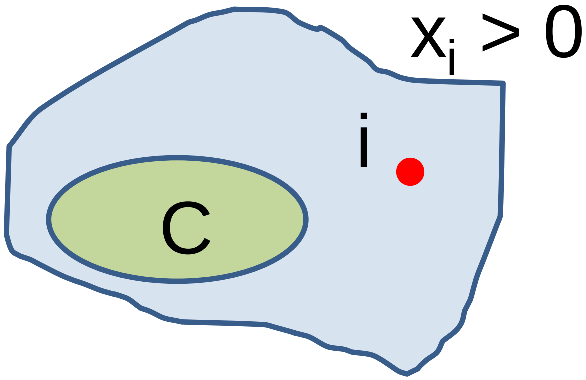
- Bondareva-Shapley theorem

Simple Games

- Definition: a game $G = (N, v)$ is simple if
  - $v(C) \in \{0, 1\}$ for any $C \subseteq N$
  - v is monotone: if $v(C) = 1$ and $C \subseteq D$, then $v(D) = 1$
- A coalition C in a simple game is said to be winning if $v(C) = 1$ and losing if $v(C) = 0$
- Definition: in a simple game, a player i is a veto player if $v(C) = 0$ for any $C \subseteq N \setminus \{i\}$
  - For monotone games, equivalent to $v(N \setminus \{i\}) = 0$
- Theorem: a simple game has a non-empty core iff it has a veto player. Further, a payoff vector $\mathbf{x}$ is in the core iff $x_i = 0$ for any non-veto player i.

Simple Games: Characterization of the Core

- Proof (<=):
  - suppose i is a veto player
  - consider a payoff vector $\underline{\mathbf{x}}$ with $x_i = 1$, $x_k = 0$ for $k \neq i$
  - no coalition C can deviate from $\underline{\mathbf{x}}$:
    - if $i \in C$, we have $\Sigma_{k \in C} x_k = 1 \geq v(C)$
    - if $i \notin C$, we have $v(C) = 0$



- Proof (=>): (no veto players)
  - consider an arbitrary payoff vector $\underline{\mathbf{x}}$:
  - we have $\Sigma_{k \in N} x_k = v(N) = 1$; thus $x_i > 0$ for some $i \in N$
  - but then some $C \subseteq N \setminus \{i\}$ can deviate:
    - since i is not a veto, $v(C) = 1$, yet $x(C) \leq x(N \setminus \{i\}) = 1 - x_i < 1$

Convex Games: Non-Emptiness of The Core

- Proposition: any convex game has a non-empty core
- Proof:
  - set $x_1 = v(\{1\})$,
    $x_2 = v(\{1, 2\}) - v(\{1\})$,
    ...
    $x_n = v(N) - v(N\backslash\{n\})$
    - i.e., pay each player his marginal contribution to the coalition formed by his predecessors
  - $\underline{\mathbf{x}}$ is a payoff vector: $x_1 + x_2 + \dots + x_n =$
    $= v(\{1\}) + v(\{1, 2\}) - v(\{1\}) + \dots + v(N) - v(N\backslash\{n\}) = v(N)$
  - remains to show that $(x_1, x_2, \dots, x_n)$ is in the core

Convex Games Have Non-Empty Core

- Proof (continued):
  - $x_1 = v(\{1\})$, $x_2 = v(\{1, 2\}) - v(\{1\})$, …, $x_n = v(N)-v(N\backslash\{n\})$
  - pick any coalition $C = \{i, j, \ldots, s\}$, where $i < j < \ldots < s$
  - we will prove $v(C) \le x_i + x_j + \ldots + x_s$, i.e., C cannot deviate
  - $v(C) = v(\{i\}) + v(\{i, j\}) - v(\{i\}) + \ldots + v(C) - v(C\backslash\{s\})$
    - $v(\{i\}) = v(\{i\}) - v(\emptyset) \le v(\{1, \ldots, i-1, i\}) - v(\{1, \ldots, i-1\}) = x_i$
    - $v(\{i, j\}) - v(\{i\}) \le v(\{1, \ldots, j-1, j\}) - v(\{1, \ldots, j-1\}) = x_j$
    - ….
    - $v(C) - v(C\backslash\{s\}) \le v(\{1, \ldots, s-1, s\}) - v(\{1, \ldots, s-1\}) = x_s$
  - thus, $v(C) \le x_i + x_j + \ldots + x_s$

Linear Duality and the Bondareva-Shapley Theorem

- Consider a superadditive* TU game G=(N,v). Recall:

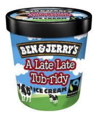
$$\text{core}(G) = \left\{ \underline{\mathbf{x}} \;\middle|\; \begin{array}{ll} x_i \geq 0 \quad \text{for any i in N}, & \\ \Sigma_{i \in N}\, x_i = v(N), & \text{(all value is allocated)} \\ \Sigma_{i \in C}\, x_i \geq v(C) \text{ for any } C \subseteq N & \text{(no deviations)} \end{array} \right\}$$

- These are all linear constraints!
- Thus the core is defined by a linear program

Example

 C: \$8,  M: \$3,  P: \$4

 w = 400 p = \$7



w = 650 p = \$9

w = 1000 p = \$13

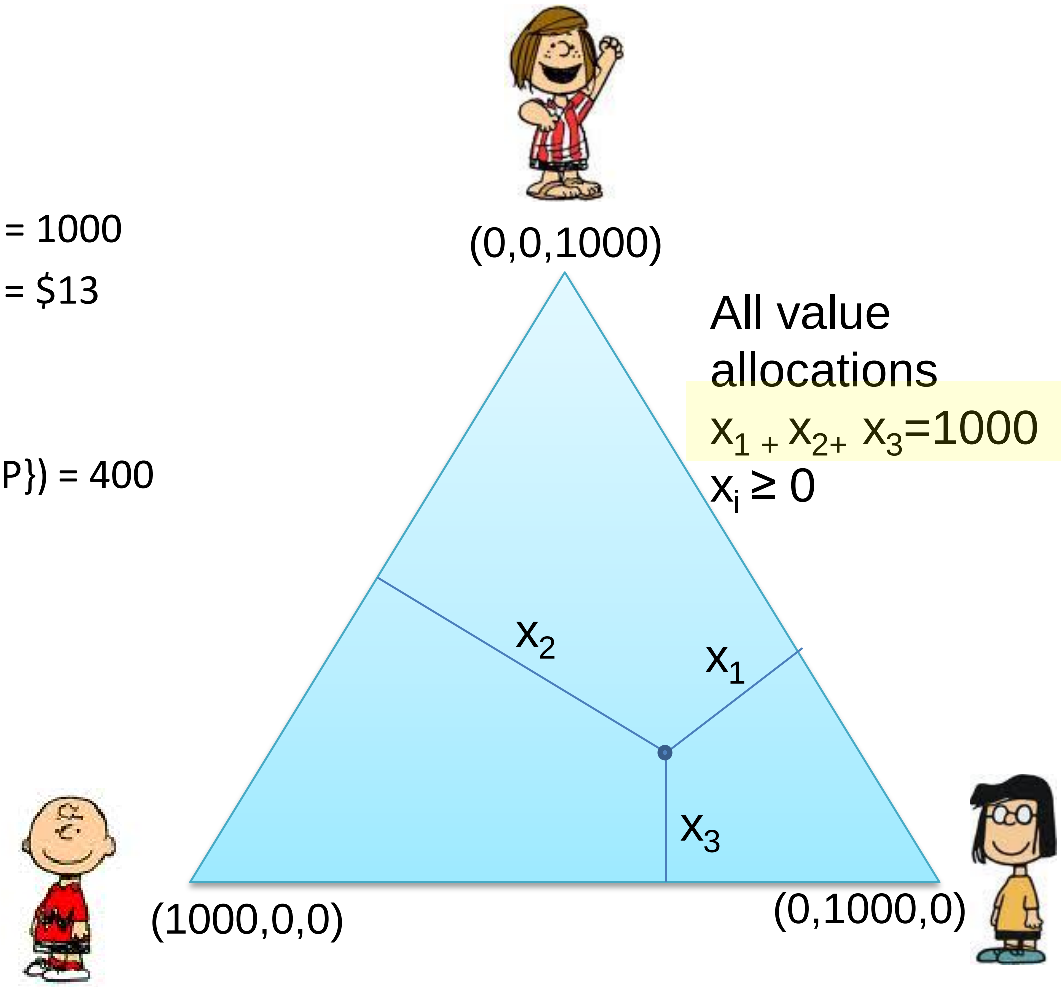
- $v(\emptyset) = v(\{M\}) = v(\{P\}) = 0$, $v(\{C\}) = 400$
- $v(\{C, M\}) = 650$, $v(\{C, P\}) = 650$, $v(\{M, P\}) = 400$
- $v(\{C, M, P\}) = 1000$

Example

 C: \$8,  M: \$3,  P: \$4

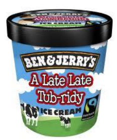
 w = 400 p = \$7  w = 650 p = \$9  w = 1000 p = \$13

- v(Ø) = v({M}) = v({P}) = 0, v({C}) = 400
- v({C, M}) = 650, v({C, P}) = 650, v({M, P}) = 400
- v({C, M, P}) = 1000

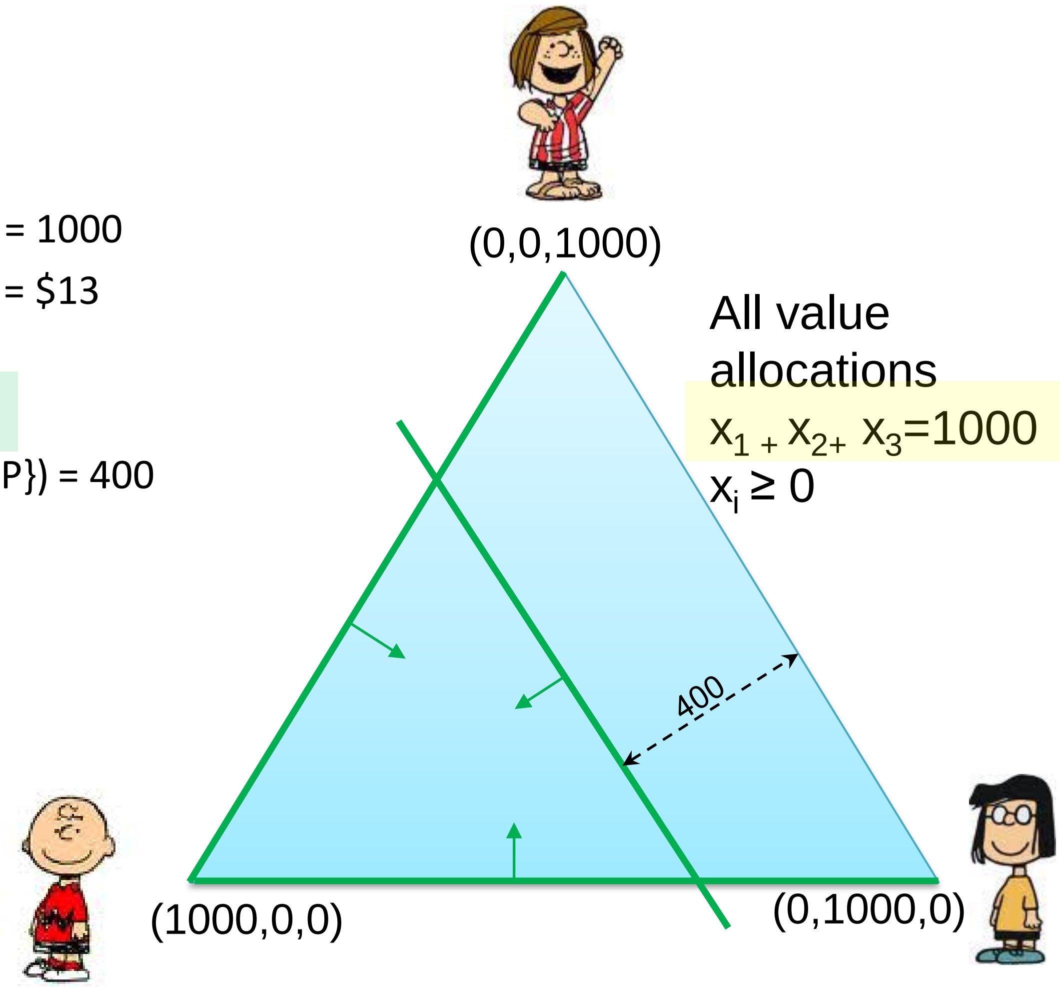


Example

 C: \$8,  M: \$3,  P: \$4

 w = 400, p = \$7  w = 650, p = \$9  w = 1000, p = \$13

- v(Ø) = v({M}) = v({P}) = 0, v({C}) = 400
- v({C, M}) = 650, v({C, P}) = 650, v({M, P}) = 400
- v({C, M, P}) = 1000

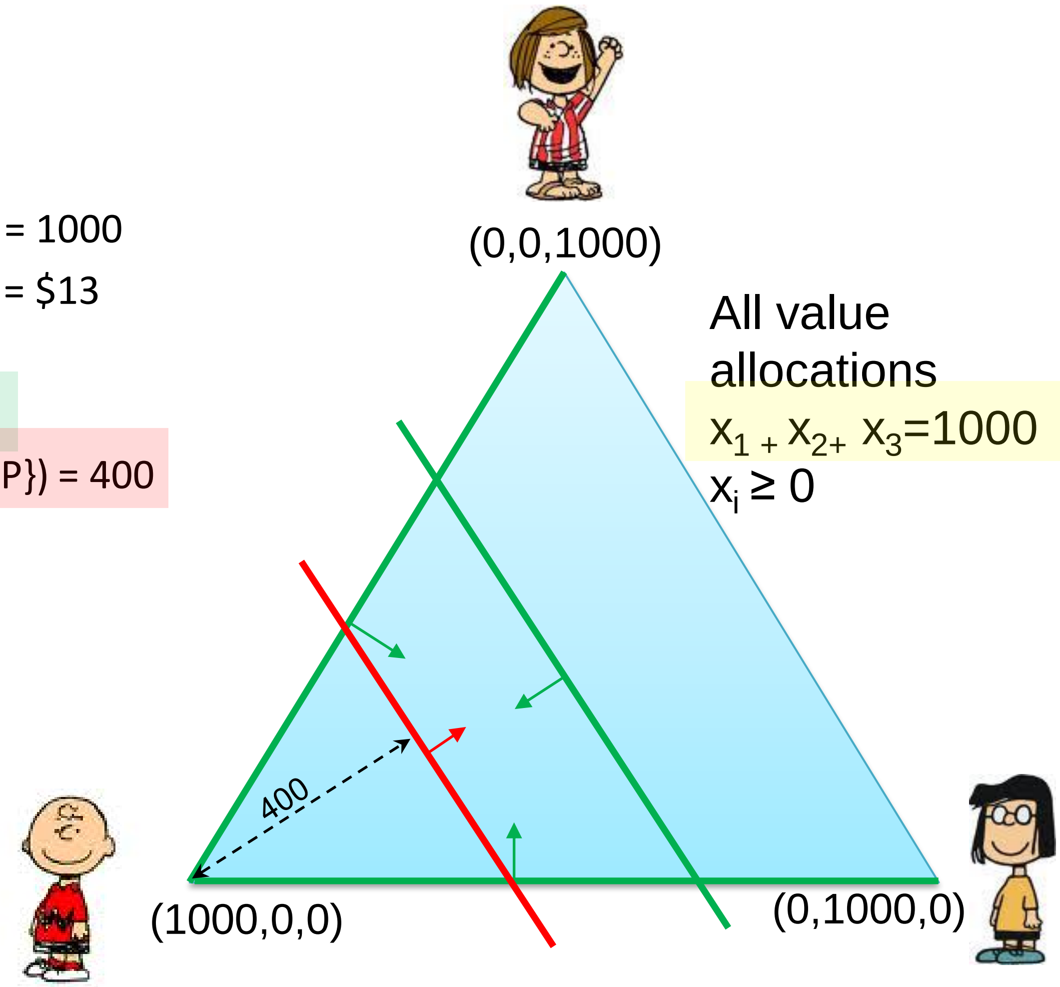


Example

 C: \$8,  M: \$3,  P: \$4

 w = 400, p = \$7  w = 650, p = \$9  w = 1000, p = \$13

- v(Ø) = v({M}) = v({P}) = 0, v({C}) = 400
- v({C, M}) = 650, v({C, P}) = 650, v({M, P}) = 400
- v({C, M, P}) = 1000

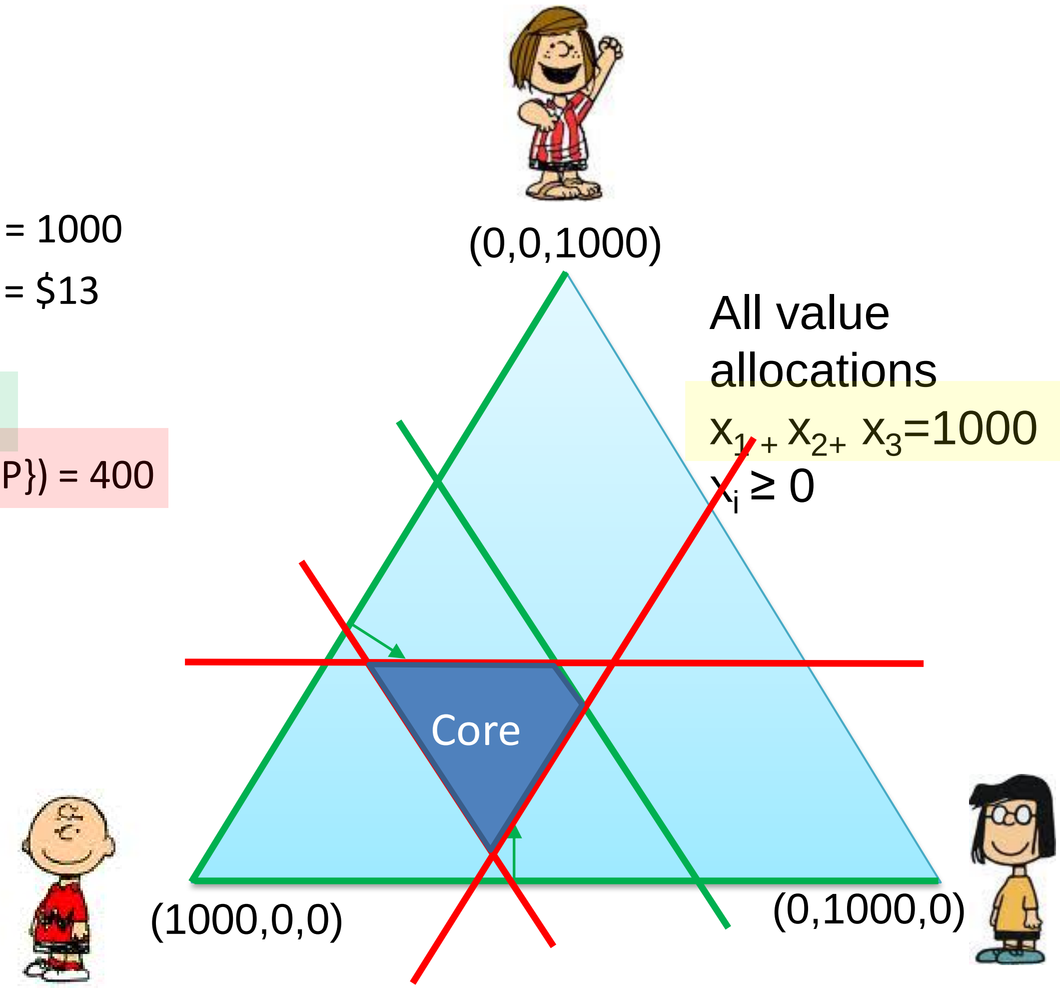


Example

 C: \$8,  M: \$3,  P: \$4

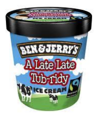
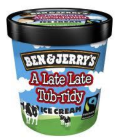
 w = 400 p = \$7  w = 650 p = \$9  w = 1000 p = \$13

- v(Ø) = v({M}) = v({P}) = 0, v({C}) = 400
- v({C, M}) = 650, v({C, P}) = 650, v({M, P}) = 400
- v({C, M, P}) = 1000

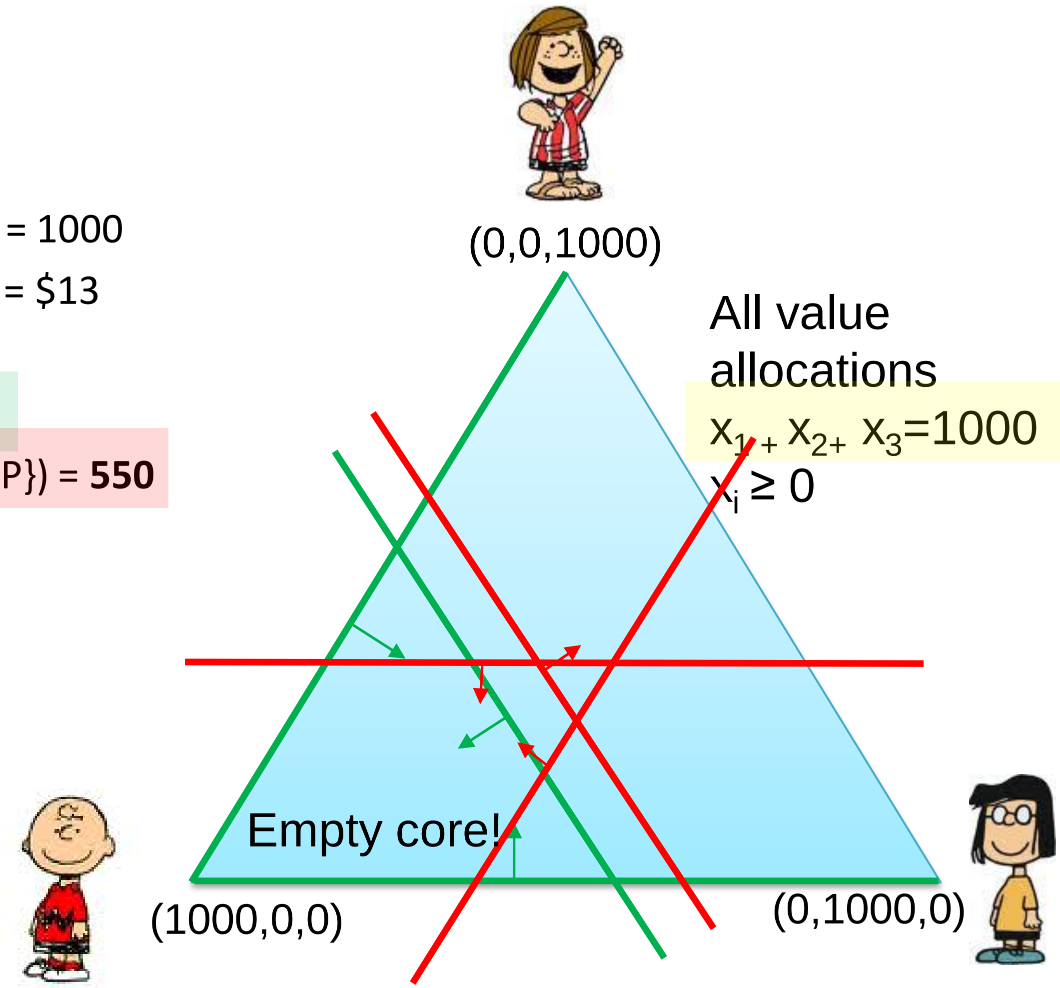


Example

 C: \$8,  M: \$3,  P: \$4

 **w = 550** p = \$7  w = 650 p = \$9  w = 1000 p = \$13

- v(Ø) = v({M}) = v({P}) = 0, v({C}) = 400
- v({C, M}) = 650, v({C, P}) = 650, v({M, P}) = **550**
- v({C, M, P}) = 1000



Linear Duality and the Bondareva-Shapley Theorem

- Consider a superadditive* TU game G=(N,v). Recall:

$$\text{core}(G) = \left\{ \underline{\mathbf{x}} \;\middle|\; \begin{array}{ll} x_i \geq 0 \quad \text{for any } i \text{ in } N, & \\ \Sigma_{i \in N}\, x_i = v(N), & \text{(all value is allocated)} \\ \Sigma_{i \in C}\, x_i \geq v(C) \text{ for any } C \subseteq N & \text{(no deviations)} \end{array} \right\}$$

- These are all linear constraints!
- Thus the core is defined by a linear program
- The dual program provides a characretization

Linear Duality and the Bondareva-Shapley Theorem

- Consider a weight vector over coalitions

$$\alpha : 2^N \ \setminus \ \emptyset \ \rightarrow [0,1]$$

A weight vector α is *balanced* if

$$\forall i \in N \ \ \sum_{C:i\in N} \alpha_C = 1$$

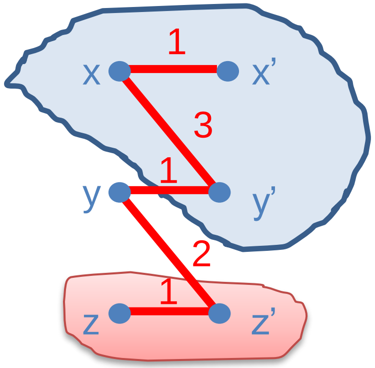
<u>BS Theorem</u>: The core of (N,v) is nonempty iff for every balanced weight vector

$$\sum_{C\subseteq N} \alpha_C \mathrm{v}(C) \leq \mathrm{v}(N)$$

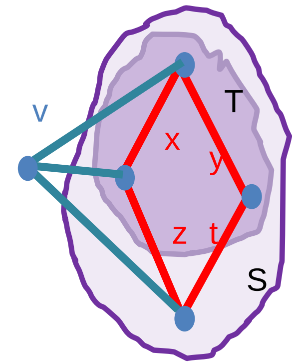
Linear Duality and the Bondareva-Shapley Theorem

- How to prove that the core of a given game is nonempty?
  - Show an allocation in the core

- How to prove that the core of a given game is empty??
  - Show a (small) balanced weight vector that violates the condition
  - (not shown here: small witnesses exist)



Games on graphs
[Myerson'77]



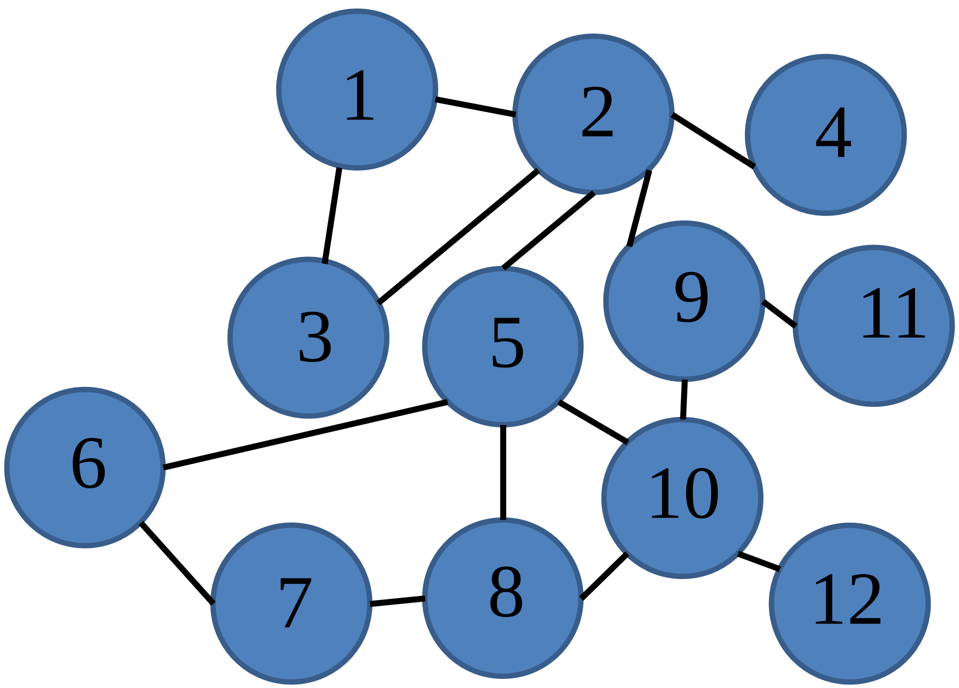
- Consider Induced subgraph games and Assignment games
- In both representations there is a graph (N,E), and

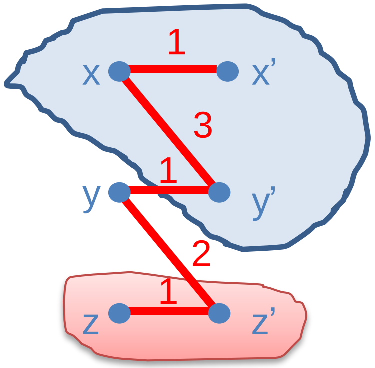
v(S U T) = v(S) + v(T) if S,T are *disconnected* in (N,E)

- For any graph H=(N,E) and any game G, we can define a game $G|_H$.
- A coalition S is valid in $G|_H$ only if S is connected in H.
- Otherwise $v|_H(S)=0$.

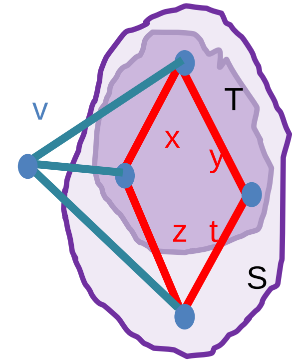
Restricted cooperation - example

- The coalition $\{2,9,10,12\}$ is allowed
- The coalition $\{3,6,7,8\}$ is not allowed





Games on graphs



- For any graph H=(N,E) and any game G, we can define a game $G|_H$.

Theorem [Demange'04]: if H=(N,E) is a **tree**, and the game G is **superadditive**, then G has a non-empty core

- Extended to general graphs in [Meir, Zick, Elkind, Rosenschein'13]

Part III: Fairness

Marginal Contribution

- A fair payment scheme would reward each agent according to his contribution
- First attempt: given a game $G = (N, v)$, set $x_i = v(\{1, ..., i\text{-}1, i\}) - v(\{1, ..., i\text{-}1\})$
  - payoff to each player = his marginal contribution to the coalition of his predecessors
- We have $x_1 + ... + x_n = v(N)$
  - $\underline{\mathbf{x}}$ is a payoff vector
- However, payoff to each player depends on the order
- $G = (N, v)$
  - $N = \{1, 2\}$, $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$
  - $x_1 = v(1) - v(\emptyset) = 5$, $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$

Average Marginal Contribution

- Idea: to remove the dependence on ordering, can average over all possible orderings
- $G = (N, v)$
  - $N = \{1, 2\}$, $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = 5$, $v(\{1, 2\}) = 20$
  - 1, 2: $x_1 = v(1) - v(\emptyset) = 5$, $x_2 = v(\{1, 2\}) - v(\{1\}) = 15$
  - 2, 1: $y_2 = v(2) - v(\emptyset) = 5$, $y_1 = v(\{1, 2\}) - v(\{2\}) = 15$
  - $z_1 = (x_1 + y_1)/2 = 10$, $z_2 = (x_2 + y_2)/2 = 10$
  - the resulting outcome is fair!
- Can we generalize this idea?

Shapley Value

- Reminder: a permutation of $\{1,\dots, n\}$ is a one-to-one mapping from $\{1,\dots, n\}$ to itself
  - let $P(N)$ denote the set of all permutations of N
- Let $S_\pi(i)$ denote the set of predecessors of i in $\pi \in P(N)$

$S_\pi(i)$	i	...

- For $C \subseteq N$, let $\delta_i(C) = v(C \cup \{i\}) - v(C)$
- <u>Definition</u>: the Shapley value of player i in a game $G = (N, v)$ with $|N| = n$ is

$$\phi_i(G) = 1/n! \sum_{\pi:\, \pi \in P(N)} \delta_i(S_\pi(i))$$

- In the previous slide we have $\phi_1 = \phi_2 = 10$

Shapley Value: Probabilistic Interpretation

- ϕ_i is i's average marginal contribution to the coalition of its predecessors, over all permutations
- Suppose that we choose a permutation of players uniformly at random, among all possible permutations of N
  - then ϕ_i is the expected marginal contribution of player i to the coalition of his predecessors

Ice-Cream Game: Shapley Value


C: \$6

M: \$4

P: \$2

w = 500
p = \$7

w = 750
p = \$9

= 1000
= \$11

- C, M, P: $\delta_C(S_\pi(C)) = 0$
- C, P, M: $\delta_C(S_\pi(C)) = 0$
- M, C, P: $\delta_C(S_\pi(C)) = 750$
- P, C, M: $\delta_C(S_\pi(C)) = 500$
- M, P, C: $\delta_C(S_\pi(C)) = 1000$
- P, M, C: $\delta_C(S_\pi(C)) = 1000$

Charlie's Shapley value:
$3250/6 \approx 542$g

Shapley Value: Properties (1)

- Proposition: in any game G, $\phi_1 + \ldots + \phi_n = v(N)$
  - $(\phi_1, \ldots, \phi_n)$ is a payoff vector for the grand coalition
- Proof:

for a permutation π, let π_i denote player in position i. Then

$\Sigma_{i=1, \ldots, n} \phi_i =$

$1/n! \, \Sigma_{i=1, \ldots, n} \Sigma_{\pi: \pi \in P(N)} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))] =$

$1/n! \, \Sigma_{\pi: \pi \in P(N)} \Sigma_{i=1, \ldots, n} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))] =$

$1/n! \, \Sigma_{\pi: \pi \in P(N)} [v(\{\pi_1\}) - v(\emptyset) + \ldots + v(N) - v(N \setminus \{\pi_n\})] =$

$1/n! \, \Sigma_{\pi: \pi \in P(N)} v(N) =$

$v(N)$

$\boxed{\pi_1 \ \pi_2 \ \ldots \ \pi_n}$

Shapley Value: Properties (2)

- Definition: a player i is a null player in a game $G = (N, v)$ if $v(C) = v(C \cup \{i\})$ for any $C \subseteq N$
- Proposition: if a player i is a null player in a game $G = (N, v)$ then $\phi_i = 0$
- Proof: if i is a null player, all summands in $\sum_{\pi: \pi \in P(N)} [\, v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))]$ equal 0
  - converse is only true if the game is monotone:
    - $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 1$, $v(\emptyset) = v(\{1, 2\}) = 0$
    - $\phi_1 = \phi_2 = 0$, but 1 and 2 are not null players

Shapley Value: Properties (3)

- Definition: given a game $G = (N, v)$, two players i and j are said to be symmetric if $v(C \cup \{i\}) = v(C \cup \{j\})$ for any $C \subseteq N \setminus \{i, j\}$
- Proposition: if i and j are symmetric then $\phi_i = \phi_j$
- Proof sketch:
  - given a permutation π, let π' denote the permutation obtained from π by swapping i and j
  - mapping $\pi \rightarrow \pi'$ is a one-to-one mapping
  - we can show $\delta_i(S_\pi(i)) = \delta_j(S_{\pi'}(j))$

$$\boxed{\ldots \quad i \ldots j \quad \ldots} \Rightarrow \boxed{\ldots \quad j \ldots i \quad \ldots}$$

Shapley Value: Properties (4)

- Definition: Let $G_1 = (N, u)$ and $G_2 = (N, v)$ be two games with the same set of players.
  Then $G = G_1 + G_2$ is the game with the set of players N and characteristic function w given by
  $w(C) = u(C) + v(C)$ for all $C \subseteq N$
- Proposition: $\phi_i(G_1+G_2) = \phi_i(G_1) + \phi_i(G_2)$
- Proof: $\phi_i(G_1 + G_2) =$

  $1/n!\ \Sigma_\pi[u(S_\pi(i) \cup \{i\}) + v(S_\pi(i) \cup \{i\}) - u(S_\pi(i)) - v(S_\pi(i))]$

  $= 1/n!\ \Sigma_\pi[u(S_\pi(i) \cup \{i\}) - u(S_\pi(i))] +$

  $1/n!\ \Sigma_\pi[v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))] = \phi_i(G_1) + \phi_i(G_2)$

Axiomatic Characterization

- Properties of Shapley value:
  1. Efficiency: $\phi_1 + \ldots + \phi_n = v(N)$
  2. Null player: if i is a null player, $\phi_i = 0$
  3. Symmetry: if i and j are symmetric, $\phi_i = \phi_j$
  4. Additivity: $\phi_i(G_1+G_2) = \phi_i(G_1) + \phi_i(G_2)$
- Theorem: Shapley value is the only payoff distribution scheme that has properties (1) - (4)

From Permutations to Coalitions

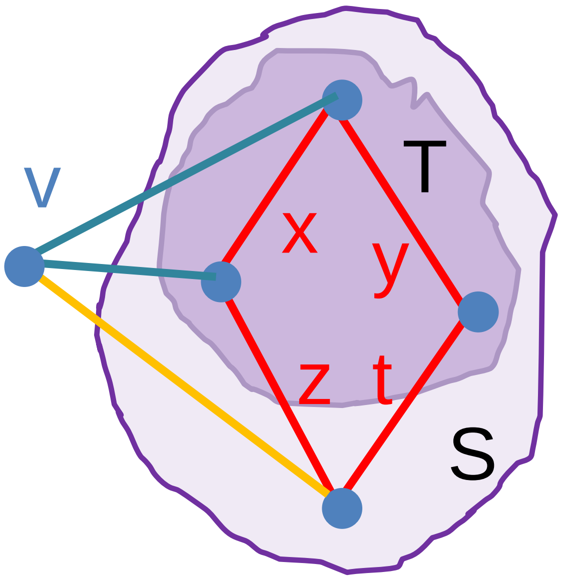
- $\phi_i(G) = 1/n! \sum_{\pi:\ \pi \in P(N)} \delta_i(S_\pi(i))$
- n! terms
- Equivalent definition:

  $\phi_i(G) = \sum_{C \subseteq N\setminus\{i\}} |C|!(n-|C|-1)!/n!\ \delta_i(C)$

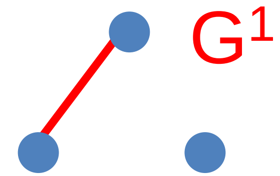
  – i appears right after C in |C|!(n-|C|-1)! permutations
- 2^{n-1} terms

Induced Subgraph Games

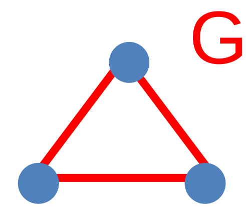
- Players are vertices of a weighted graph
- Value of a coalition = total weight of internal edges
  - $v(T) = x + y$, $v(S) = x + y + z + t$

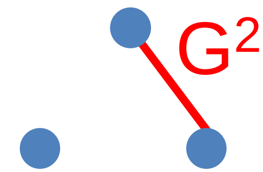


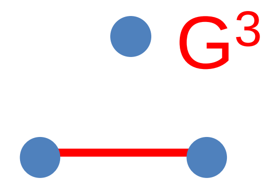
Induced Subgraph Games: Shapley Value



- Shapley value in ISGs is easy to compute:
  - let $E = \{e^1, \ldots, e^k\}$ be the list of edges of the graph
  - let G^j be the induced subgraph game on the graph that contains edge e^j only
  - we have $G = G^1 + \ldots + G^k$
  - $\phi_i(G^j) = w(e^j)/2$ if e^j is adjacent to i and 0 otherwise
  - $\phi_i(G)$ = (weight of edges adjacent to i)/2







Shapley Value in Weighted Voting Games

- In a simple game $G = (N, v)$, a player i is said to be pivotal
  - for a coalition $C \subseteq N$ if $v(C) = 0$, $v(C \cup \{i\}) = 1$
  - for a permutation $\pi \in P(N)$ if he is pivotal for $S_\pi(i)$
- In simple games player i's Shapley value = Pr[i is pivotal for a random permutation]
  - measure of voting power
- Shapley value is widely used to measure power in various voting bodies

Weighted Voting Games: UK

- United Kingdom, 2010:
  - 650 seats, $q = 326$
  - Conservatives (C): 306
  - Labour (L): 258
  - Liberal Democrats (LD): 57
  - Scottish National Party (SNP): 6
  - Democratic Unionist Party (DUP): 8
  - 6 other parties (O), with a total of 15 seats
- DUP is pivotal for {L, LD, SNP} and {C, O}
- $\phi_{DUP} = 1/(720)(3!2!+2!3!) = 1/30$

Weighted Voting Games: UK

- United Kingdom, 2010:
  - 650 seats, q = 326
  - Conservatives (C): 306
  - Labour (L): 258
  - Liberal Democrats (LD): 57
  - Scottish National Party (SNP): 6
  - Democratic Unionist Party (DUP): 8
  - 6 other parties (O), with a total of 15 seats
- L and LD are symmetric,
  so have same Shapley values

Weighted Voting Games: UK

- United Kingdom, 2015:
  - 650 seats, q = 326
  - Conservatives (C): 330
  - Labour (L): 232
  - Scottish National Party (SNP): 56
  - Liberal Democrats (LD): 8
  - Democratic Unionist Party (DUP): 8
  - 7 other parties (O), with a total of 16 seats
- C is a veto player, all others are null players
- $\phi_C = 1$, other parties' values are 0

Weighted Voting Games: UK

- United Kingdom, 2017:
  - 650 seats, $q = 326$
  - Conservatives (C): 317
  - Labour (L): 262
  - Scottish National Party (SNP): 35
  - Liberal Democrats (LD): 12
  - Democratic Unionist Party (DUP): 10
  - 9 other parties (O), with a total of 14 seats
- C is pivotal for every coalition apart from {L, SNP, LD, DUP, O} and $\emptyset$
- $\phi_C = 1/(720)(6! - 2*5!) \approx 0.67$

Weighted Voting Games: UK

- United Kingdom, 2017:
  - 650 seats, q = 326
  - Conservatives (C): 317
  - Labour (L): 262
  - Scottish National Party (SNP): 35
  - Liberal Democrats (LD): 12
  - Democratic Unionist Party (DUP): 10
  - 9 other parties (O), with a total of 14 seats
- players L, SNP, LD, DUP, O are symmetric
- $\phi_L = 1/(720)(6! - 4*5!)/5 \approx 0.06$

Computational Issues in Coalitional Games

- Problem 1: the naive **representation** of a coalitional game is exponential in the number of players n
  - need to list values of all coalitions
- Problem 2: We are usually interested in **algorithms** whose running time is polynomial in n
  - Checking stability → go over 2^n coalitions
- So what can we do?

Part IV: Representation and Computation

Overview

- Introduction
- Definitions
- The Core
- Representations of games
  - Representation types
  - Simple games
  - Characterization of the core
- Computational and algorithmic questions
- Examples in different types of games

How to Deal with Representation Issues?

- Strategy 1: oracle representation
  - assume that we have a black-box poly-time algorithm that, given a coalition $C \subseteq N$, outputs its value $v(C)$
  - Useful for proofs on general games
- Strategy 2: restricted classes
  - consider games on combinatorial structures
  - Examples: Routing games, Rescue teams
  - problem: not all games can be represented in this way
- Strategy 3: give up on worst-case succinctness
  - devise complete representation languages that allow for compact representation of interesting games
  - (next 2 slides)

Synergy Coalition Games

[Conitzer & Sandholm'06]

- Superadditive game: $v(C \cup D) \geq v(C) + v(D)$ for any two disjoint coalitions C and D
- <u>Idea</u>: if a game is superadditive, and $v(C) = v(C_1) + \ldots + v(C_k)$ for any partition $(C_1, \ldots, C_k)$ of C (no synergy), no need to store $v(C)$
- Representation: list $v(\{1\}), \ldots v(\{n\})$ and all synergies
- Succinct when there are few synergies
- This representation allows for efficient checking if an outcome is in the core.
- However, it is still hard to check if the core is non-empty.

Marginal Contribution Nets [Ieong&Shoham'05]

- Idea: represent the game by a set of rules of the form pattern → value
  - pattern is a Boolean formula over N
  - value is a number
- A rule applies to a coalition if its fits the pattern
- v(C) = sum of values of all rules that apply to C
- Example:

  R_1: $(1 \land 2) \lor 5 \rightarrow 3$

  R_2: $2 \land 3 \rightarrow -2$

  $v(\{1, 2\}) = 3$, $v(\{2, 3\}) = -2$, $v(\{1, 2, 3\}) = 1$