Simple and Optimal Mechanism in Efficiency and Equality Trade-off

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Mechanism Design

Single objective

- Profit
  - Myerson optimal mechanism
- Social welfare
  - Vickrey-Clarke-Groves mechanism

……

Public Resource Allocation

Public Resource
- Supply-demand imbalance
- Heterogeneous agents

Multi-objective mechanism design
- Efficiency
- Fairness

A good mechanism in practice
- ✓Easy implementation

Efficiency

Social

Social welfare maximization

Individual

Market equilibrium

Fairness

Fairness defined based on utility

Fairness defined based on allocation (non utility-based)

Utility-based Fairness

- Envy-free: everyone likes his more than others
- Egalitarian: same utility for everyone
- Proportional fairness: utility proportional to contribution
- Max-min: always maximize the minimum utility

Opportunity Fairness (non utility-based)

- Defined only on allocation (independent of utility)
- Lottery

Problem 1: Vehicle License Allocation

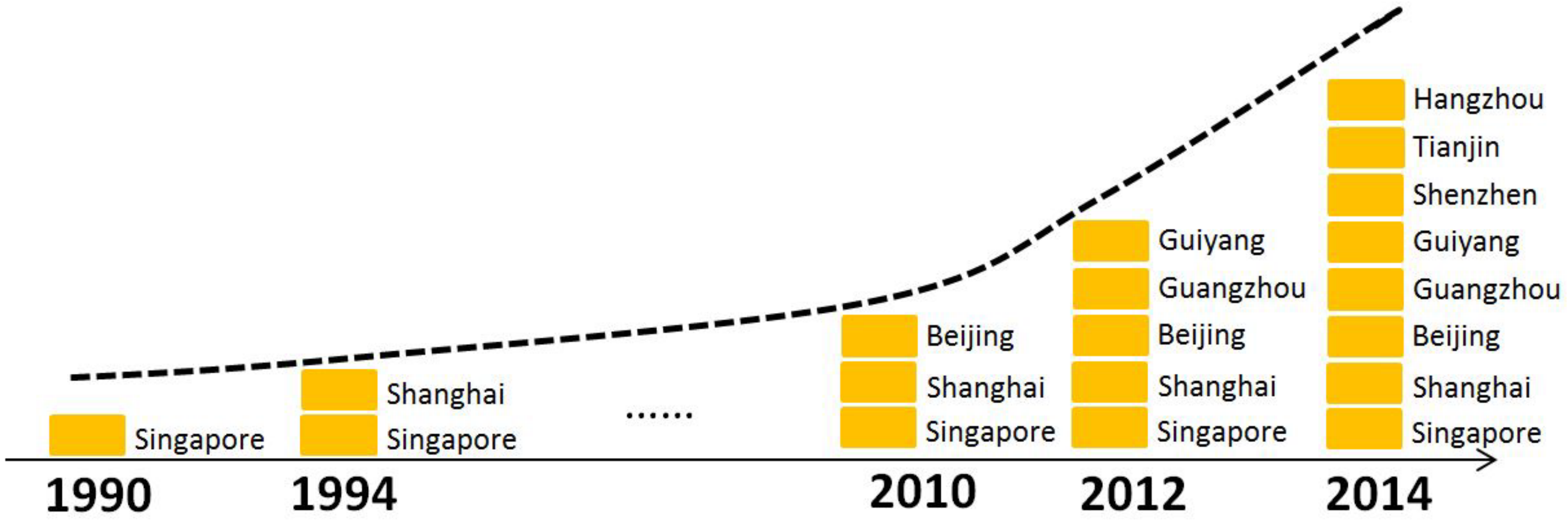
(Chen, Qi, Wang WINE'17)
(Chen, Qi, Wang & Wang '22)

Background : many big cities started the vehicle licenses control due to the terrible traffic and air quality problem.

FIGURE: Heavy traffic on a highway during the morning rush hours in Shanghai

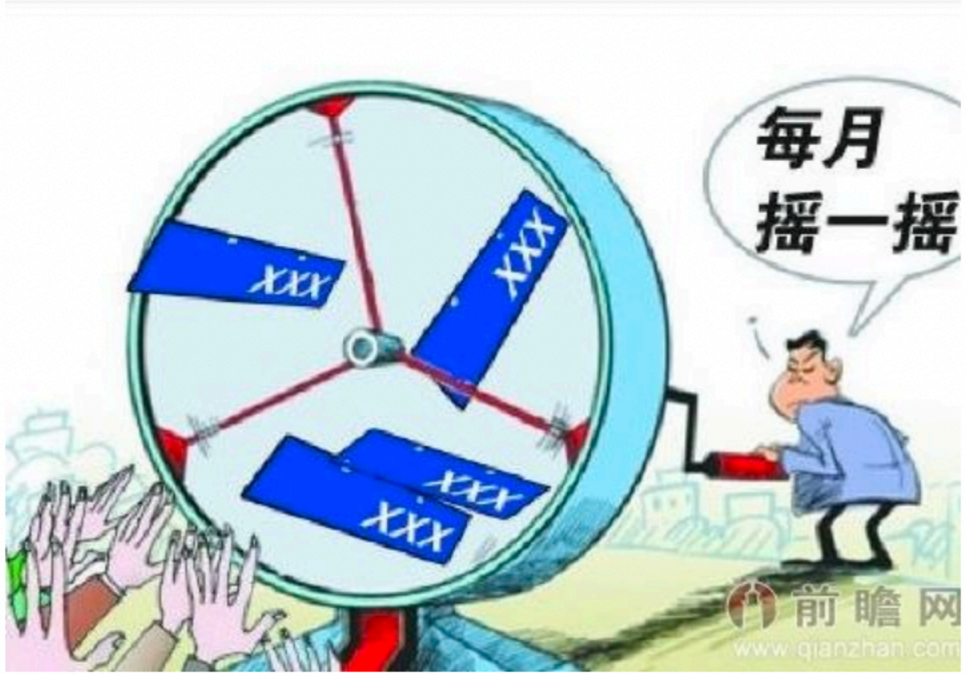
FIGURE: Beijing citizen wearing industrial strength mask in a smog day

Who Did?



In each time period, only a small number of plates can be allocated to the potential car buyers.

Vehicle licenses allocation is a tough problem



Efficiency

Current Mechanism 1: Auction

- Representative: Shanghai(1994-2013.3), Singapore(1990-now)
- Pros: efficiency
- Cons: low affordability, unfair for poor

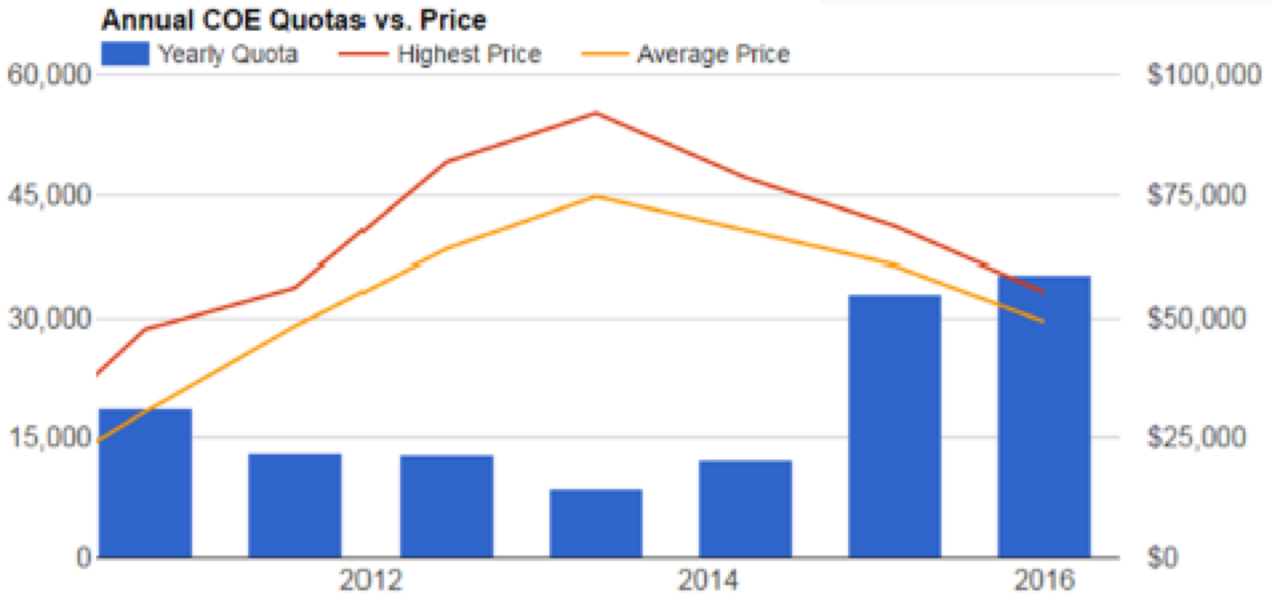
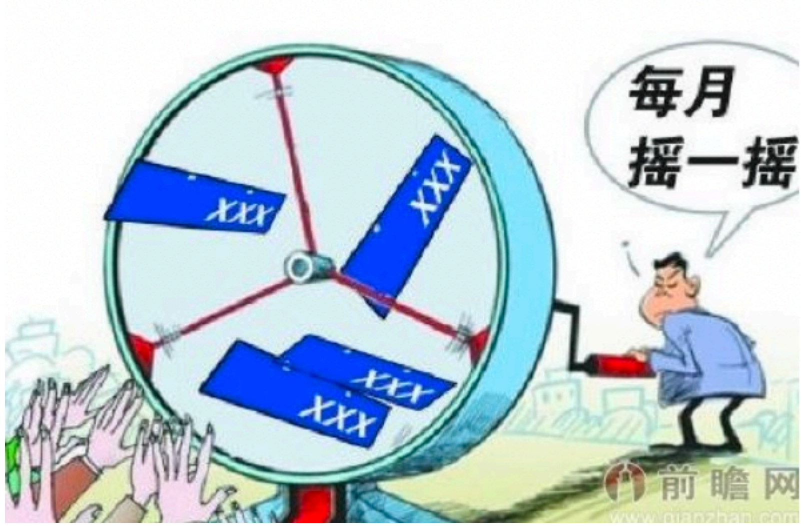


FIGURE: Singapore's Auction ($92100 in 2013)

Equality



Current Mechanism 2: Lottery

Representative: Beijing(2010-now), Guiyang(2011-now)

- Pros: fairness
- Cons: low efficiency and wining probability, no value exploration

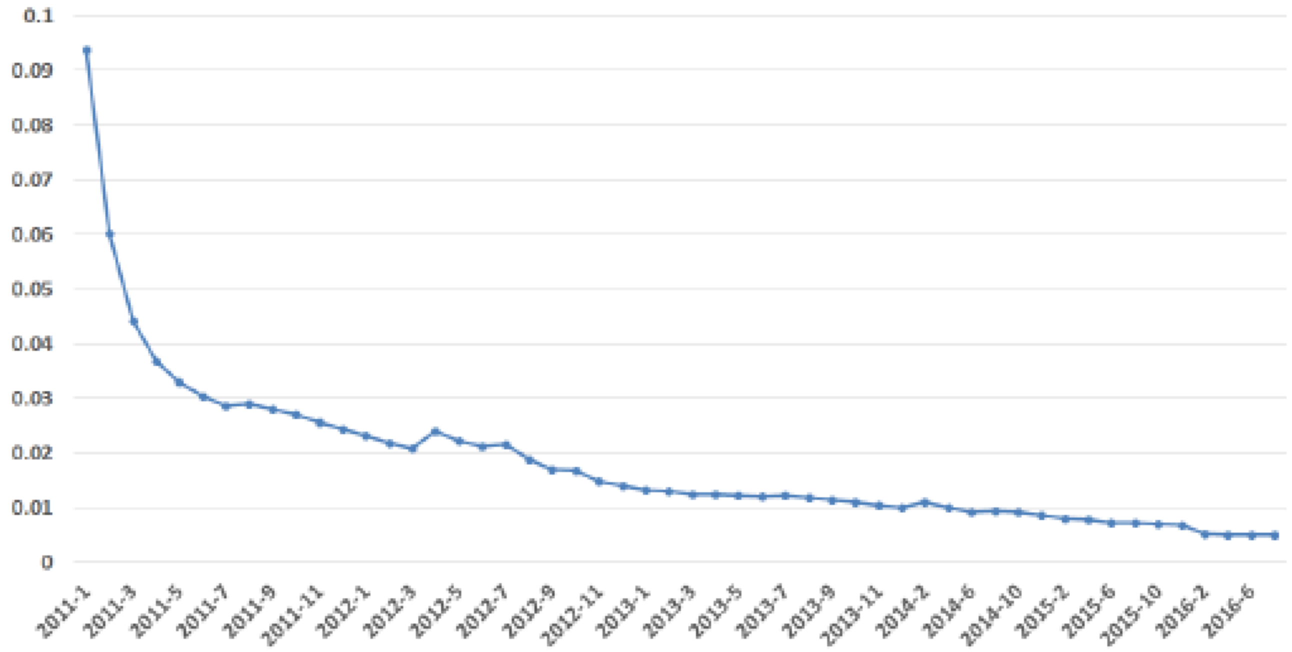
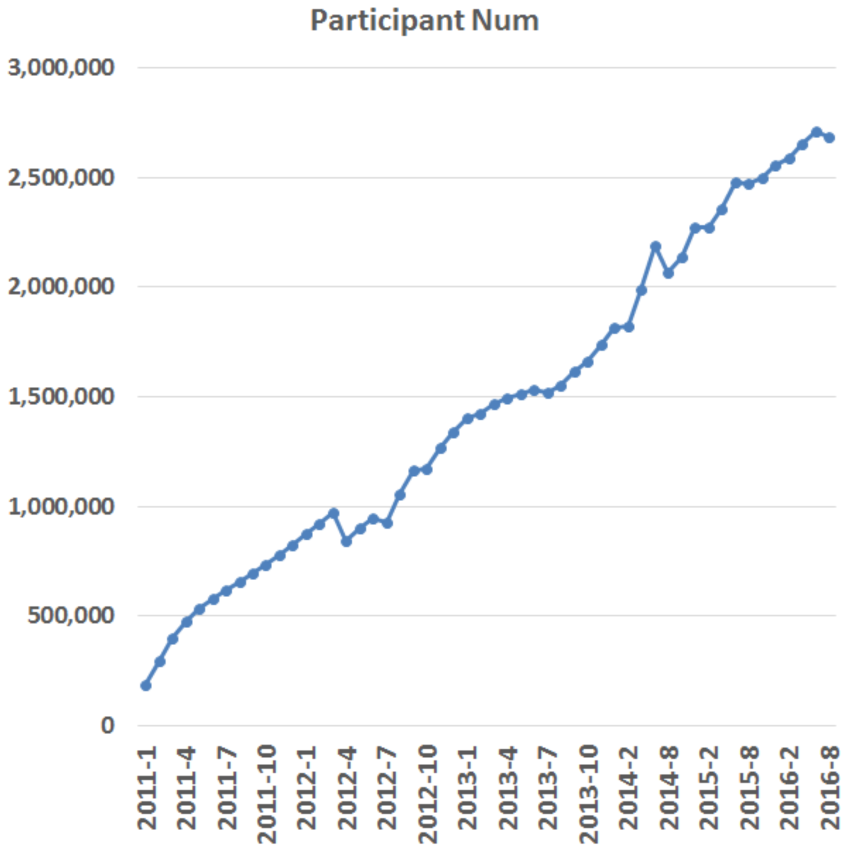
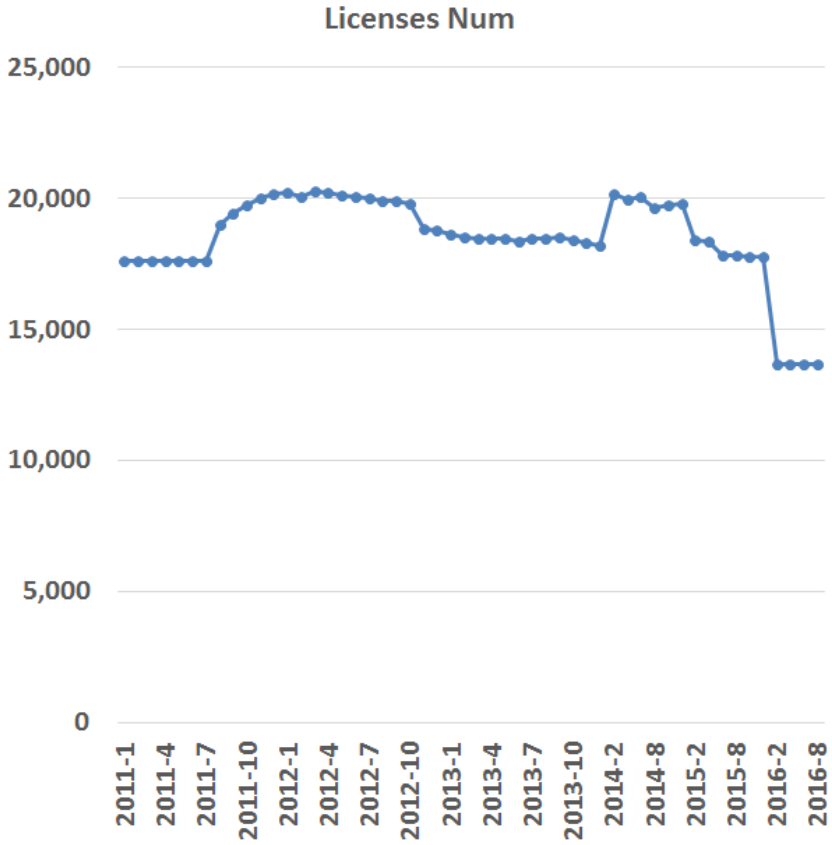


FIGURE: Winning Probability in Beijing's Lottery(0.501% in June,2016)

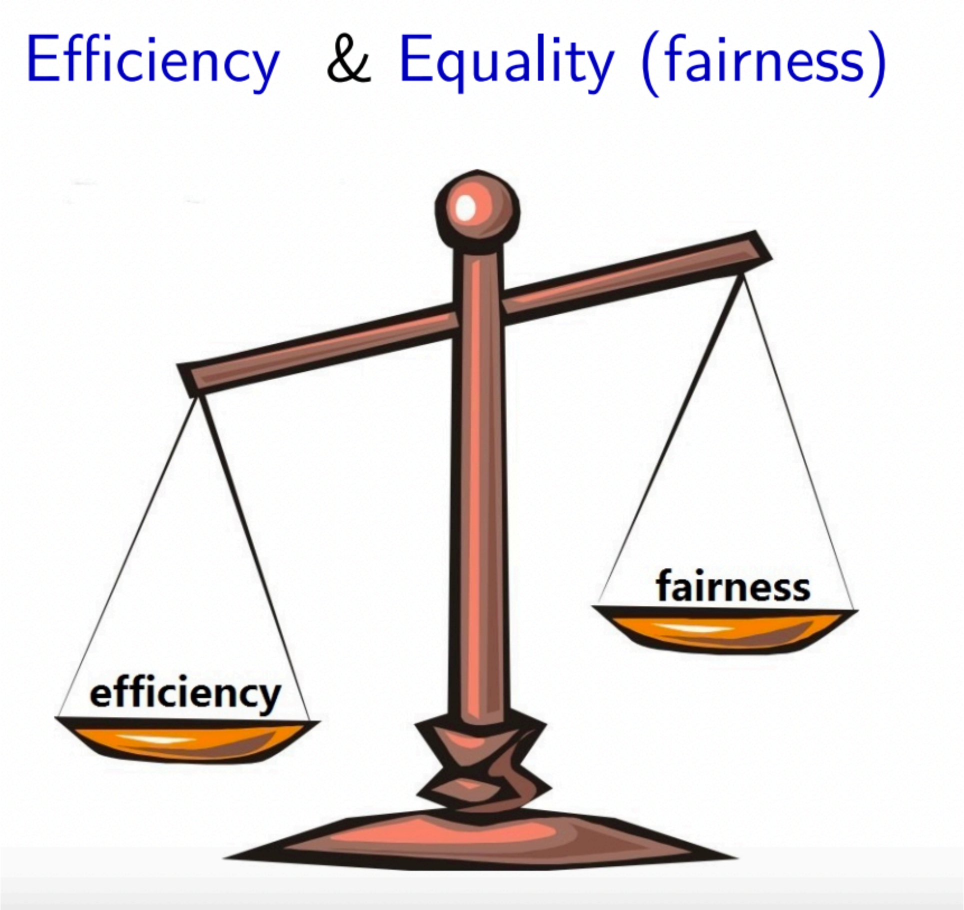
Beijing's Data

Waiting time: over 30 years!



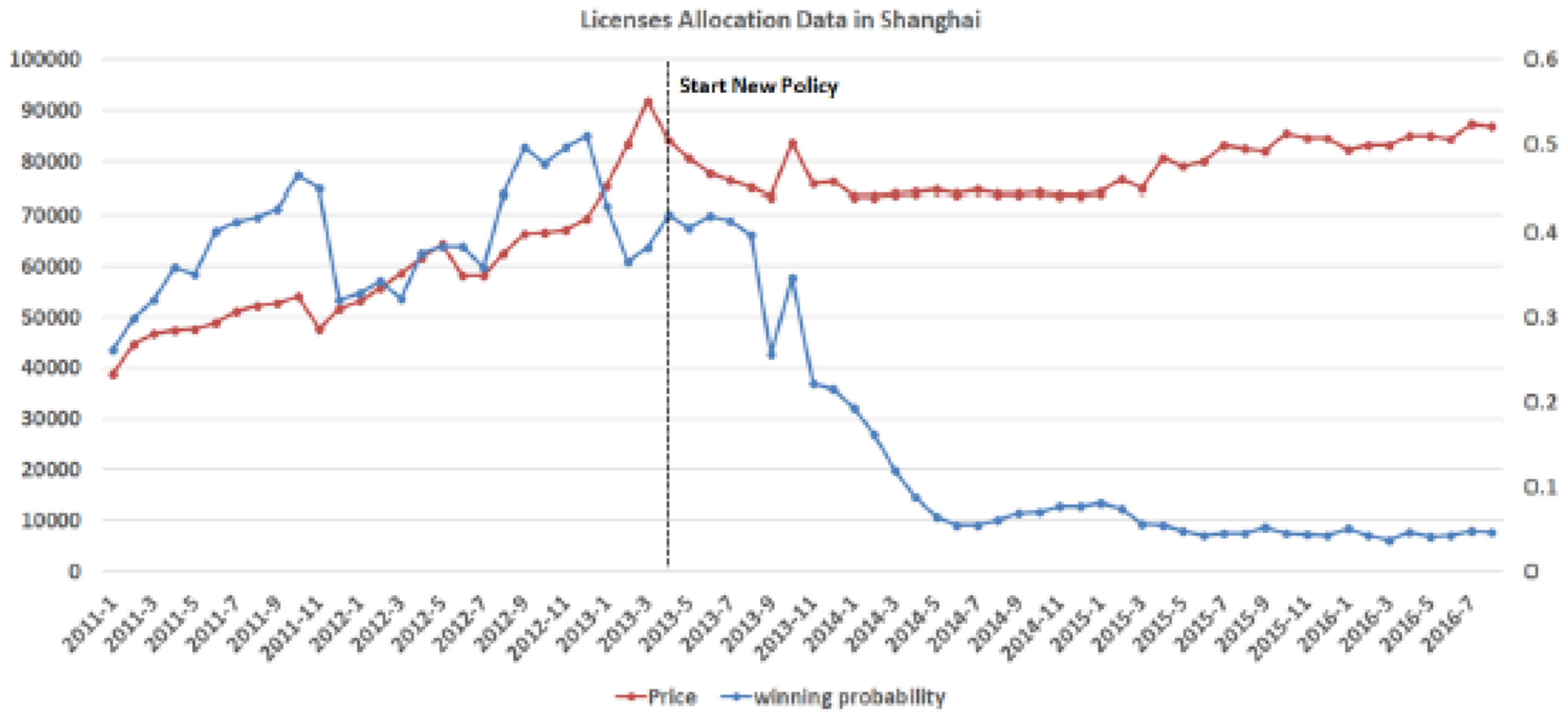


Make a Balance



Current Mechanism 3: Reserve-price Lottery

Representative City: Shanghai(2013.4-now)
Pros: easy implementation (winners in lottery pay the reserve price)
Cons: hard to set price, no exploration, inefficiency



Current Mechanism 4: Simultaneous Auction and Lottery

- Representative City: Guangzhou, Hangzhou, Shenzhen...
- Each player chooses either auction or lottery first.
  - Pros: consider both fairness and efficiency
  - Cons: hard to compute the equilibrium, untruthful

Which one is optimal?

- Auction 
- Lottery 
- Reserve-price Lottery 
- Simultaneous Auction and Lottery  + 

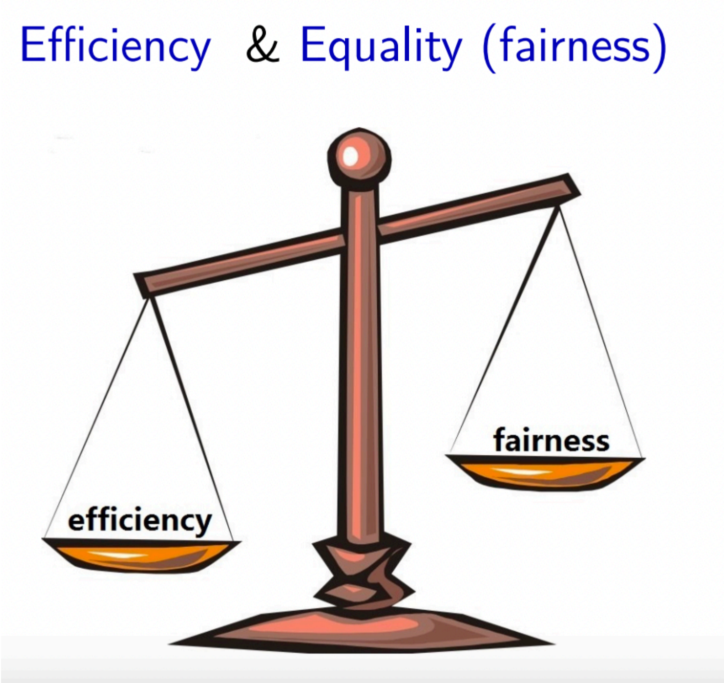
Key Result

First auction then lottery (AtL), is the best choice!

- ✓Efficiency
- ✓Equality
- ✓Easy implementation

The Problem

- How to optimally allocate k homogeneous goods to n (> k) unit-demand players with efficiency & equality trade-off?



The Model

- k homogeneous goods, n unit-demand players;
- Suppose w.l.o.g. $v_1 \geq v_2 \geq \cdots \geq v_n$ are the true private values;
- No need random & independent assumption on values;
- Maximizing efficiency while guaranteeing certain level of equality;
- Pre-computed probabilities $q_1, q_2, \ldots, q_n$ before knowing values;
- As simple as possible.

Start with Two-group Mechanism

STEP 1 The government determines an integer number $k \in [0, K]$, a ratio $\gamma \in [1, \frac{N}{K}]$.

STEP 2 Highest $k\gamma$ bids will be allocated k licenses via lottery (Lottery I);
$K - k$ licenses allocate to remaining $N - k\gamma$ players by lottery (Lottery II).

$$\underbrace{v_1 \quad v_2 \cdots v_{\gamma k}}_{\text{Lottery I: k licenses}} \quad \underbrace{v_{\gamma k+1} \cdots \quad v_N}_{\text{Lottery II: K-k licenses}}$$

Two-group Framework

MECHANISM 1: Two-Group Framework

1. The government determines an integer number $n_1 \in [0, n]$, a ratio $t_1 \in [\frac{k}{n}, \min\{\frac{k}{n_1}, 1\}]$.
2. Every player i submits a bid $b_i \geq 0$.
3. Highest n_1 bidders will be allocated with probability t_1 (Group I); The other $n - n_1$ players will be allocated the remaining goods with probability $\frac{k-n_1 t_1}{n-n_1}$ (Group II).
4. Only winners of Group I pay $p = \frac{nt_1-k}{(n-n_1)t_1} \cdot b_{(n_1+1)}$; others pay $p = 0$.

Theorem

This mechanism is incentive compatible and ex-post individual rational.

Efficiency Measure

Efficiency is defined as the social welfare:

$$Z(\mathbf{q}) = \sum_{i=1}^{n} q_i v_i$$

The efficiency of our mechanism is

$$Z(n_1, t_1) = t_1 \sum_{i=1}^{n_1} v_i + \frac{k - n_1 t_1}{n - n_1} \sum_{i=n_1+1}^{n} v_i$$

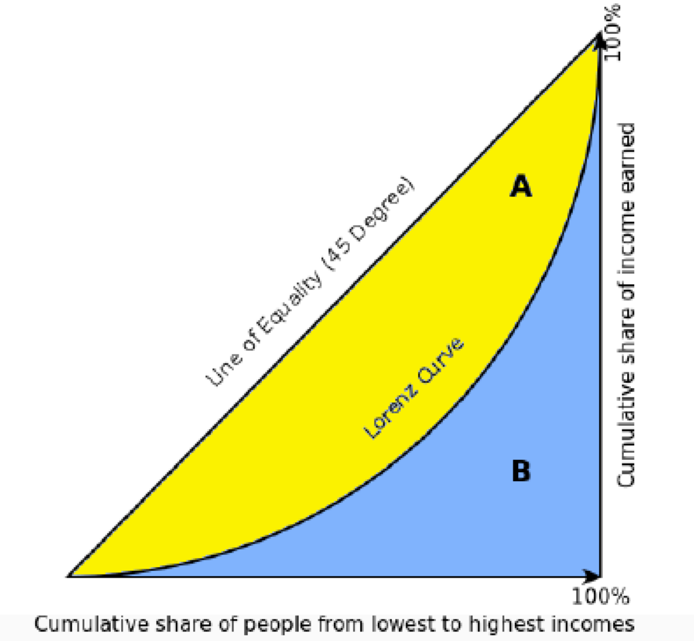
Equality

Equality is defined by Gini coefficient [Corrado, 1936]

$$\eta(\mathbf{q}) = 1 - G(\mathbf{q})$$

The equality of our mechanism is

$$\eta(n_1, t_1) = \frac{n(k - n_1 t_1) + n_1 k}{nk}$$



- Gini index: G=A/(A+B)
- Equality=1-G

A Unified Framework

Proposition

By choosing different parameters k and γ, our mechanism either includes the existing mechanisms as special cases or outperforms them.

- Auction: $n_1 = k, t_1 = 1,$
- Lottery: $t_1 = \frac{k}{n}$ (or $n_1 = 0$),
- Reserved-price Lottery: Setting $n_1 = n_p, t_1 = \frac{k}{n_p}$
- Simultaneous Auction and Lottery: Setting $n_1 = \ell, t_1 = 1$

Question

How to compute optimal mechanism under the framework?

Math Formula of optimal two-group mechanism

Let $v_1 \geq v_2 \geq \cdots \geq v_N$ be the true private values. Our problem is to solve the following programming:

$$
\begin{aligned}
\max_{n_1, t_1} \quad & Z(n_1, t_1) = t_1 \sum_{i=1}^{n_1} v_i + \frac{k - n_1 t_1}{n - n_1} \sum_{i=n_1+1}^{n} v_i \\
\text{s.t.} \quad & \eta(n_1, t_1) = \frac{n(k - n_1 t_1) + n_1 k}{nk} \geq c, \\
& \frac{k}{n} \leq t_1 \leq 1, \\
& 1 \leq n_1 \leq n, \\
& n_1 t_1 \leq k.
\end{aligned}
$$

where $c \in [0, 1]$ is a pre-determined constant to measure the least required equality.

Challenges

Determine n_1, t_1

before knowing any information about the values v_i

Slight change, Big difference

Suppose $n = 6, k = 3$, and $\eta(k, \gamma) \geq c = 5/6$, then:

- $(v_1, v_2, v_3, v_4, v_5, v_6) = (4, 3.5, 3, 1, 0.8, 0.6)$, then the optimal mechanism is with $n_1 = 3$ and $t_1 = 2/3$.
- If $(v_1, v_2, v_3, v_4, v_5, v_6) = (5, 3.5, 3, 1, 0.8, 0.6)$, then the optimal mechanism is with $n_1 = 1$ and $t_1 = 1$.
- If $(v_1, v_2, v_3, v_4, v_5, v_6) = (4, 3.5, 3, 3, 0.8, 0.6)$, then the optimal mechanism is with $n_1 = 4$ and $t_1 = 5/8$.

Main Result: Optimal mechanism

(Chen, Qi, Wang & Wang Recent work)

Theorem (1)

If $\Delta_i \geq \Delta_{i+1}$, $\forall\ i = 0, \ldots, n-1$, where $\Delta_i := v_i - v_{i+1}$, then $\forall c$, the optimal n_1, t_1 can be computed quickly with $n_1 = \frac{nk(1-c)}{n-k}$, $t_1 = 1$ (i.e., Auction then Lottery).

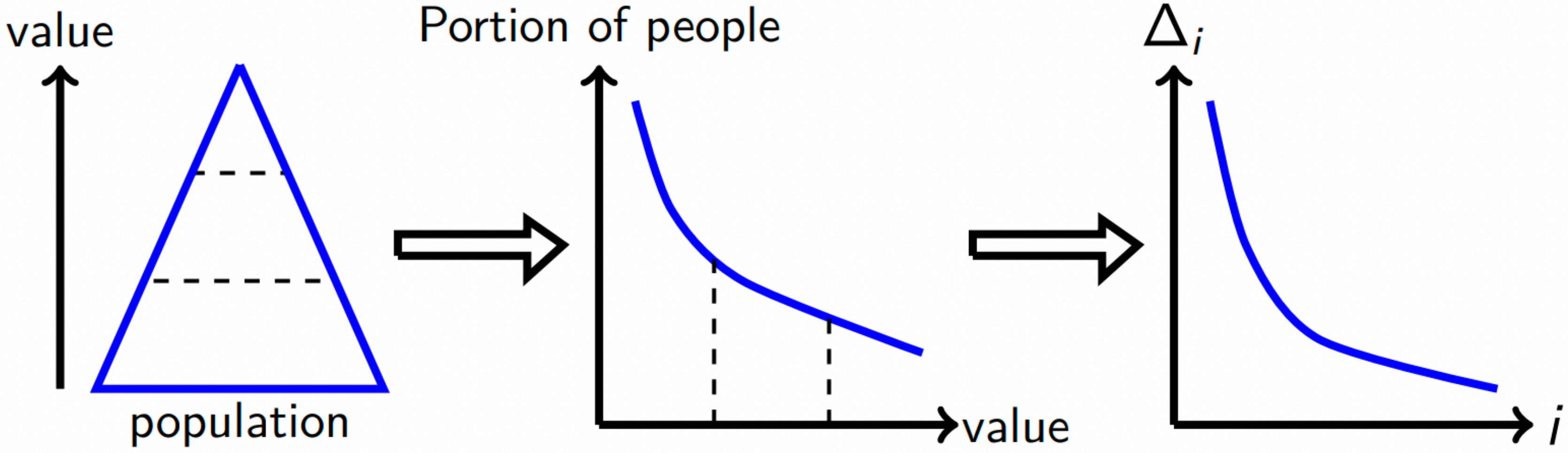


Figure: pyramid-shaped value lead to decreasing spacing

Main Result: Optimal mechanism

Theorem (2)

If $\Delta_i \geq \Delta_{i+1}$, $\forall\ i \leq \frac{n-1}{2}$, $\Delta_i \geq \Delta_{n-i}$, $\forall\ i \leq \frac{n}{2}$, *and* $n \geq 2k$, *then* $\forall c$, *the optimal* n_1, t_1 *can be computed quickly with* $n_1 = \frac{nk(1-c)}{n-k}$, $t_1 = 1$ *(i.e., Auction then Lottery).*

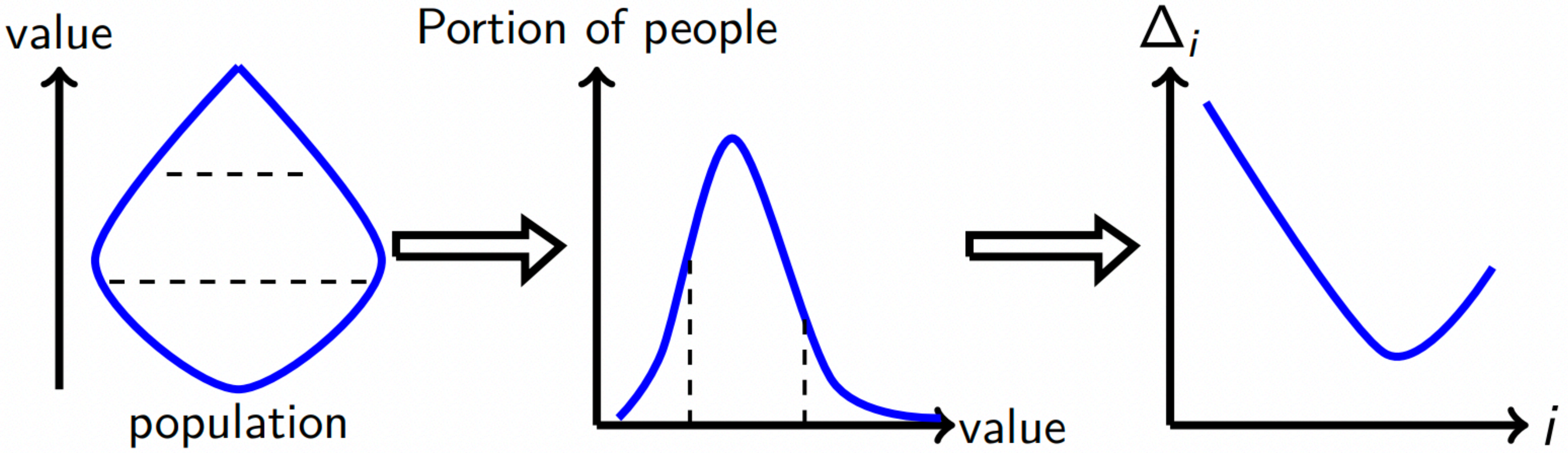


Figure: olive-shaped value negatively skewed spacing

Random Values

(Chen, Qi & Wang WINE'17)

- Assume players' values v are i.i.d. from F (f)

Random Values

Assume players' values v are i.i.d. from F (f)

Theorem

If the probability density function $f(\cdot)$ is non-increasing, then every Pareto optimal mechanism is Auction then Lottery.

Theorem

If $f(\cdot)$ satisfies $f(F^{-1}(x)) \geq f(F^{-1}(1-x)), \forall x \in [0, 1/2]$, and $f(F^{-1}(1-x))$ is non-decreasing about $x \in [0, 1/2]$, then every Pareto optimal mechanism is Auction then Lottery.

Main Result

Surprisingly!

For all the common distributions, the optimal mechanism is:

$$\gamma = 1, k = \frac{NK(1-c)}{N-K}$$ (first auction, then lottery!).

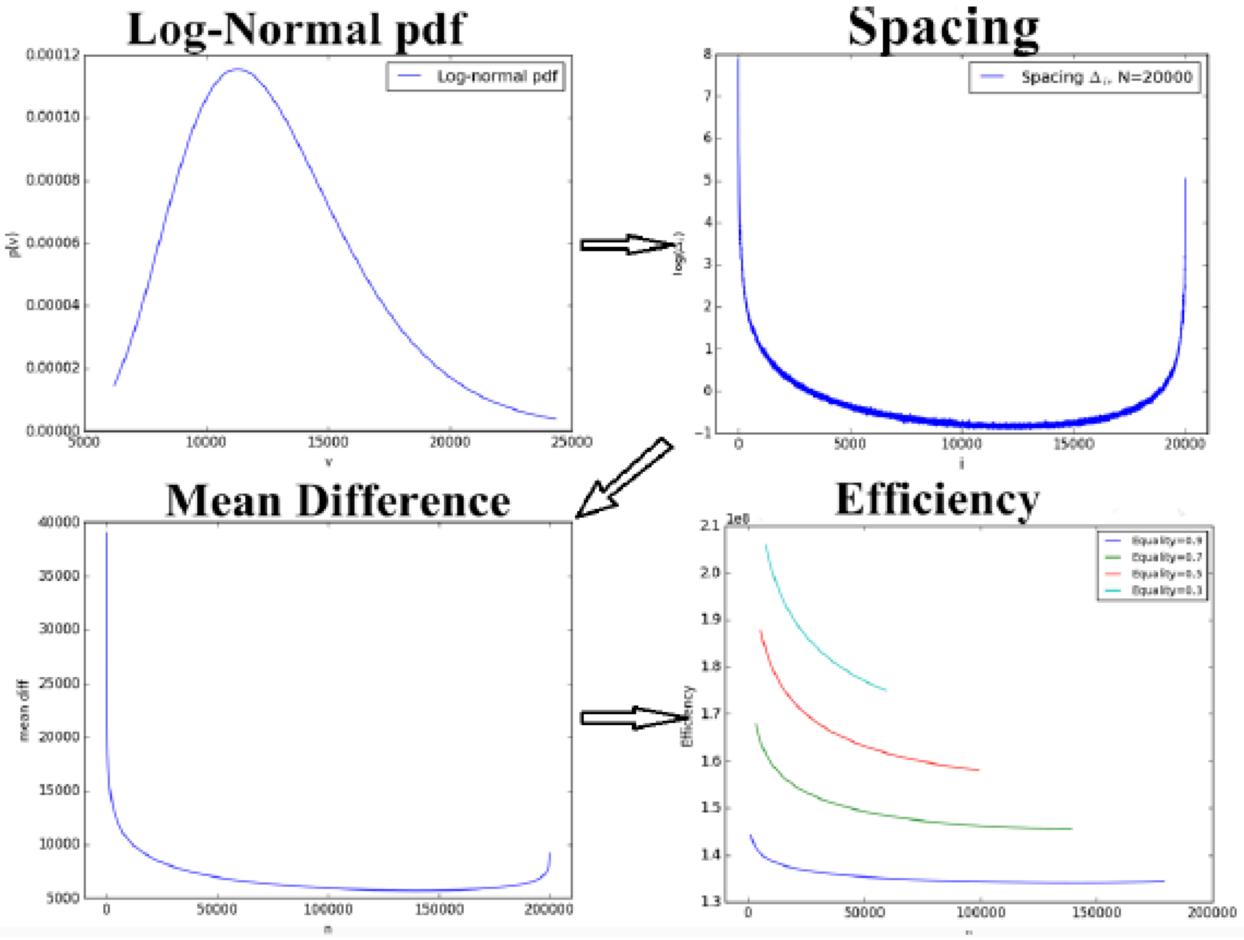
THEORETICAL Classic distributions: Normal, Uniform, Exponential

EMPIRICAL Power law relationship(economics and empirical study): Log-normal, Power-law, Truncated normal, gamma

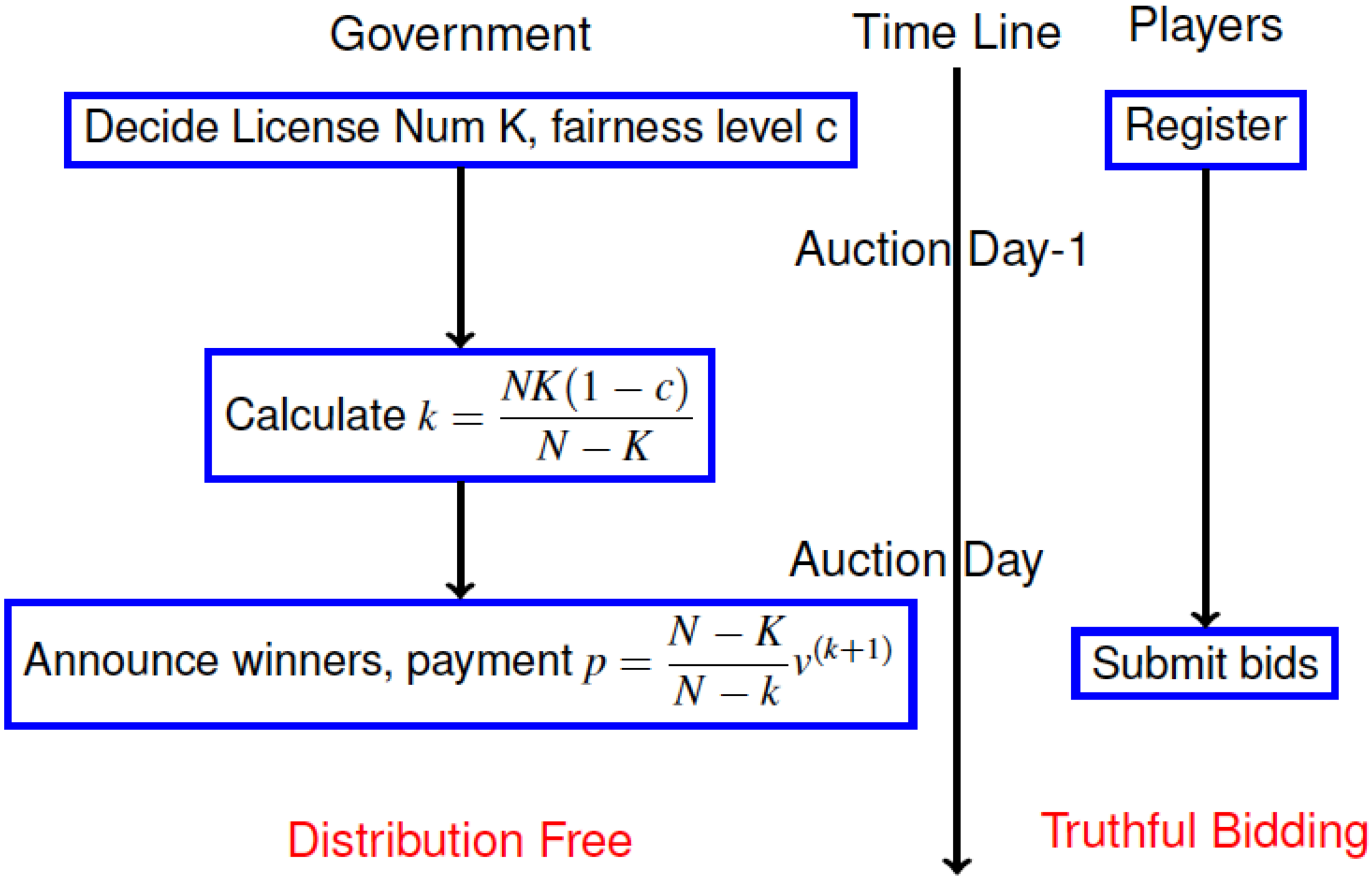
- Semi-distribution free!
- No need to know any information about the distribution

Key Ideas

efficiency with fairness constraint $\Leftarrow$ *mean difference* $\Leftarrow$ *spacing* $\Leftarrow$ *pdf information*



Procedure of Our Optimal Mechanism



Good?

Mechanism ”Auction then Lottery” is always Pareto optimal when:

1. there are two group of probabilities;
2. the values v_i satisfy some natural conditions.

What about its general performance?

Benchmark: The real optimal solution

- Values $v_1 \geq v_2 \geq \cdots \geq v_n$ can be any case;
- The allocation probabilities can be mutually different.

$$\max_{\mathbf{q}} \quad Z(\mathbf{q}, \mathbf{v}) := \sum_{i=1}^{n} q_i v_i \tag{1a}$$

$$\text{s.t.} \quad \frac{1}{nk}\left(\sum_{i=1}^{n}(2i-1)q_i\right) \geq c, \tag{1b}$$

$$\sum_{i=1}^{n} q_i = k, \tag{1c}$$

$$1 \geq q_1 \geq q_2 \geq \ldots \geq q_n \geq 0. \tag{1d}$$

Approximation Ratio

- For any given equality level c;
- Denote $\mathbf{q}^{al}$ as the allocation probabilities of AtL mechanism with $t_1 = 1, n_1 = \frac{nk(1-c)}{n-k}$;

$$\text{Approximation Ratio} := \min_{\mathbf{v}\in V} \frac{Z(\mathbf{q}^{al},\mathbf{v})}{\max_{\mathbf{q}\in\mathcal{Q}} Z(\mathbf{q},\mathbf{v})} = \min_{\mathbf{v}\in V}\min_{\mathbf{q}\in\mathcal{Q}} \frac{Z(\mathbf{q}^{al},\mathbf{v})}{Z(\mathbf{q},\mathbf{v})}$$

Note: $$\min_{\mathbf{v}\in V}\min_{\mathbf{q}\in\mathcal{Q}} \frac{Z(\mathbf{q}^{al},\mathbf{v})}{Z(\mathbf{q},\mathbf{v})} = \min_{\mathbf{q}\in\mathcal{Q}}\min_{\mathbf{v}\in V} \frac{Z(\mathbf{q}^{al},\mathbf{v})}{Z(\mathbf{q},\mathbf{v})}$$

Main Result: Approximate Optimality

(Chen, Qi, Wang & Wang Recent work)

Theorem

For every c, using mechanism AtL can always guarantee at least $\frac{3}{4}$ of the optimal social welfare, i.e.,

$$\textit{Approximation ratio} = \min_{\mathbf{v}\in V_1} \frac{Z(\mathbf{q}^{al}, \mathbf{v})}{\max_{\mathbf{q}\in\mathcal{Q}} Z(\mathbf{q}, \mathbf{v})} \geq \frac{3}{4}$$

Better Approximation at Certain Conditions

Theorem

If $\Delta_1 \geq \Delta_2 \cdots \geq \Delta_{n-1}$*, then* $\forall c$*, the efficiency of mechanism AtL*

$$Z(\mathbf{q}^{al}, \mathbf{v}) \geq \max_{\mathbf{q} \in \mathcal{Q}} Z(\mathbf{q}, \mathbf{v}) - v_1, \forall \mathbf{v} \in V$$

Better Approximation at Certain Conditions

Theorem

If $\Delta_1 \geq \Delta_2 \cdots \geq \Delta_{n-1}$, *then* $\forall c$, *the efficiency of mechanism AtL*

$$Z(\mathbf{q}^{al}, \mathbf{v}) \geq \max_{\mathbf{q} \in \mathcal{Q}} Z(\mathbf{q}, \mathbf{v}) - v_1, \ \forall \mathbf{v} \in V$$

Theorem

If $\exists s \geq \frac{n}{2}$ *s.t.* $\Delta_1 \geq \Delta_2 \geq \ldots \geq \Delta_s$, $\Delta_s \leq \Delta_{s+1} \leq \ldots \leq \Delta_n$ *and* $\Delta_{s-i} \geq \Delta_{s+i}, \forall i \leq \min\{s, n-s\}$, *then* $\forall c$, *the efficiency of mechanism AtL*

$$Z(\mathbf{q}^{al}, \mathbf{v}) \geq (1 - \frac{k}{n}) \max_{\mathbf{q} \in \mathcal{Q}} Z(\mathbf{q}, \mathbf{v}) - 3v_{max}, \ \forall \mathbf{v} \in V$$

Simulation: Comparison under distribution

(a) Efficiency Under $ln\mathcal{N}(9.6228, 0.2129)$, $n = 400000$, $k = 8000$

	Z_{al}	$\mathcal{E}_{opt}$	Gap
c = 0.3	183618004	183624500	0.0035%
c = 0.5	168916563	168925666	0.0054%
c = 0.8	144402231	144406293	0.0028%
c = 0.9	135013621	135017294	0.0027%

Table: Comparison Under Different Distributions

Comparison under distribution

(a) Efficiency Under $TN(16802, 8184^2)$

	Z_{al}	Z_{opt}	Gap
c = 0.3	253859011	253867192	0.0032%
c = 0.5	224153384	224171696	0.0082%
c = 0.8	175932376	175941577	0.0052%
c = 0.9	158118046	158134262	0.0103%

Table: Comparison Under Different Distributions

Comparison under distribution

(a) Efficiency Under $Exp(16802)$

	Z_{al}	Z_{opt}	Gap
c = 0.3	539854494	540019060	0.0305%
c = 0.5	445652648	445745078	0.0207%
c = 0.8	282866045	282921361	0.0196%
c = 0.9	217811469	217853933	0.0195%

(b) Efficiency Under $\mathcal{U}(2627, 30977)$

	Z_{al}	Z_{opt}	Gap
c = 0.3	213797147	213799878	0.0013%
c = 0.5	191117159	191119972	0.0015%
c = 0.8	157096389	157096481	0.0000%
c = 0.9	145754486	145755179	0.0005%

Table: Comparison Under Different Distributions

Comparison under different values

	Max Gap	Min Gap	Average Gap
n= 10000,k = 100,c=0.9, $\mathcal{U}(0,1)$	0.396%	0.091%	0.147%
n= 10000,k = 100,c=0.5, $\mathcal{U}(0,1)$	1.302%	0.333%	0.477%
n= 10000,k = 100,c=0.3, $\mathcal{U}(0,1)$	1.125%	0.411%	0.560%
n= 10000,k = 200,c=0.5, $\mathcal{U}(0,1)$	0.889%	0.013%	0.016%
n= 10000,k = 400,c=0.5, $\mathcal{U}(0,1)$	0.941%	0.053%	0.197%
n= 10000,k = 800,c=0.5, $\mathcal{U}(0,1)$	0.939%	0.060%	0.195%
n= 10000,k = 1600,c=0.5, $\mathcal{U}(0,1)$	0.687%	0.013%	0.144%
n= 10000,k = 3200,c=0.5, $\mathcal{U}(0,1)$	0.543%	0.013%	0.110%
n= 10000,k = 1000,c=0.5, $ln\mathcal{N}(0,1)$	0.041%	0.035%	0.038%
n= 10000,k = 1000,c=0.5, $ln\mathcal{N}(0,0.5)$	0.032%	0.030%	0.031%
n= 10000,k = 1000,c=0.5, $ln\mathcal{N}(0,0.25)$	0.026%	0.020%	0.022%

Table: Comparison for 1000 times

Comparison under different values

n=1000,k=100,c=0.5	Z_{al}	Z_{opt}	Gap
half $\mathcal{U}(0,1)$, half $\mathcal{U}(10,11)$	834.772	1048.980	20.421%
half $\mathcal{U}(1,2)$, half $\mathcal{U}(9,10)$	782.468	948.834	17.534%
half $\mathcal{U}(2,3)$, half $\mathcal{U}(8,9)$	730.829	850.529	14.074%
half $\mathcal{U}(3,4)$, half $\mathcal{U}(7,8)$	677.608	750.389	9.699%
half $\mathcal{U}(4,5)$, half $\mathcal{U}(6,7)$	626.154	650.201	3.698%
All $\mathcal{U}(5,6)$	575.894	575.895	0.193%

Table: Comparison with Two Value clusters

Problem 2: Public Housing
(Zhou, Qi, Wang, Wang WINE’17)

Private house is hardly affordable for most residents.

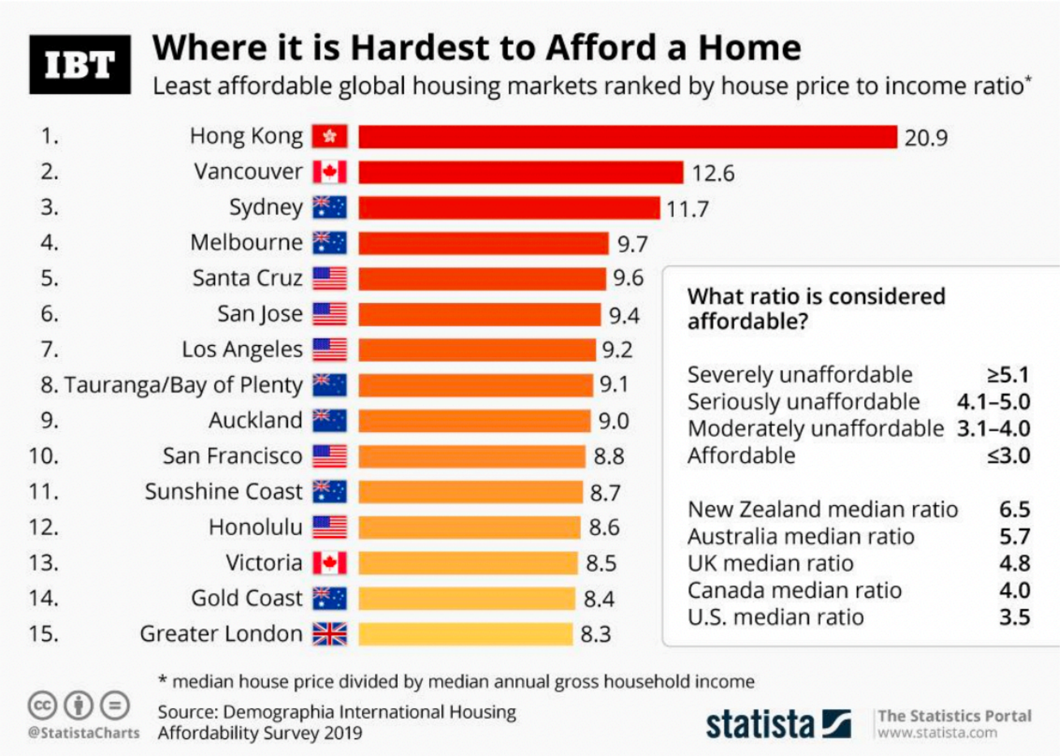


FIGURE: Median house price to annual income

Public Housing in Hong Kong

Low income households are willing to live in public housing, with very low rents.

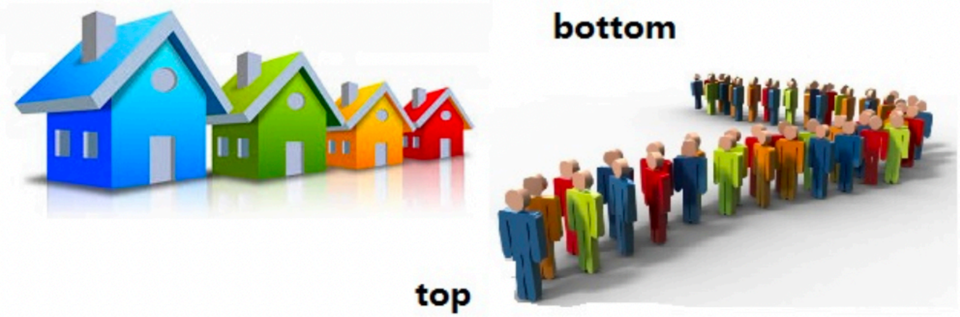
- In Hong Kong, 44.7% of the population (3.3 million people) are living in public houses.
- About 20,000 applications per year.
- Over 250,000 applications are still waiting.
- Average idle waiting time is more than 5 years.

Current Allocation Rules

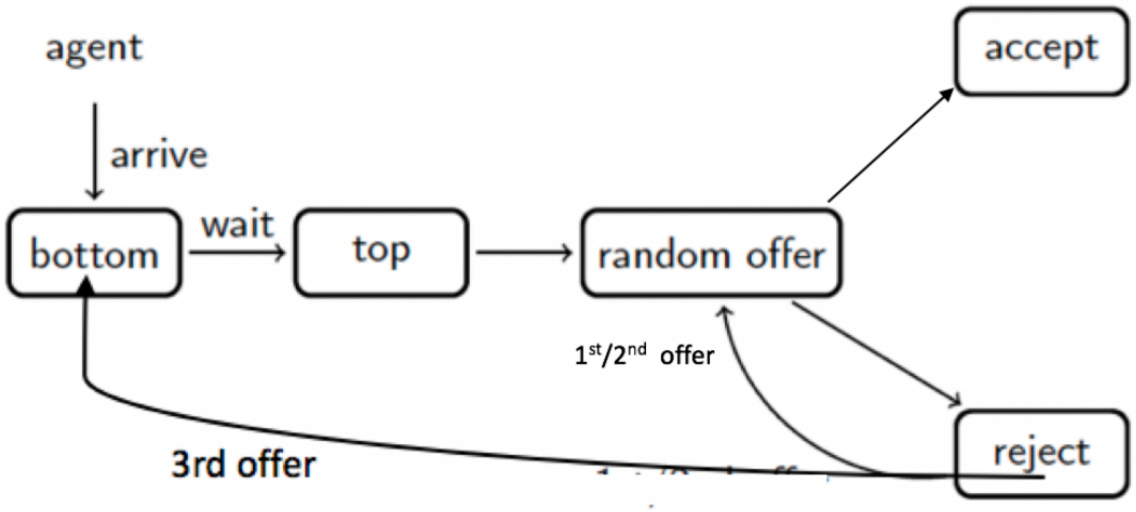
- Lottery (e.g., New York)

- single lottery
- repeated lottery

- Waitlist (e.g., Hong Kong)



Current Waitlist System in Hong Kong



Question: Will it be better if allowing other deferral number?

Objective

Previous research has shown that lottery is outcome equivalent to waitlist with specific parameter (deferral number).

Our objective: optimize the waitlist mechanism under several metrics (idle waiting time, social welfare, etc)

Our Work

We investigate a unified waitlist mechanism where agents could have various numbers of deferral chances, and derives the optimal choice.

- consider agents’ optimal strategies
- characterize the equilibrium state

Agent's Problem

Assume agents are affected by the aggregate actions of all other agents, and each agent actually faces a Markov Decision Process.

τ: idle waiting time in equilibrium state.

- State space: $S = \{-\lfloor\tau\rfloor - 1, -\lfloor\tau\rfloor, \cdots, -1, 0, 1, \cdots, k\}$.
- Action set: for $s < 0$, {wait}; for $s \geq 0$, {Accept, Reject}.
- Transition:
  - for $s < k$, matches if choosing 'Accept', and goes to state $s + 1$ otherwise;
  - for $s = k$, matches if choosing 'Accept' and goes back to state $-\lfloor\tau\rfloor$ with probability $\lfloor\tau\rfloor + 1 - \tau$ and state $-\lfloor\tau\rfloor - 1$ with probability $\tau - \lfloor\tau\rfloor$ otherwise.

Agent's Strategy

$V(s)$: value of state s
A_s: action set at state s
s_a: state after acting a at state s.

For an agent with outside option α,

$$V_\alpha(s) = \delta \cdot \mathbb{E}_v[\max_{a \in A_s}\{(v - \alpha)\mathbf{1}_{\{matched\}} + V(s_a)(1 - \mathbf{1}_{\{matched\}})\}].$$

↓ state value

↓ utility if accept the offer

↓ utility if stay to next period

Optimal Strategy: Based on a threshold on the outside option

Evaluation Metrics

- Idle waiting time τ_k.
  - number of periods taken from the bottom position to the top position.
  - smaller is better
- Match value $v_k(\alpha)$:

$$v_k(\alpha) = \frac{1}{\pi_k(\alpha)}[Pr(\text{matches } v_H \text{ house}) + v_L \cdot Pr(\text{matches } v_L \text{ house})].$$

  - $\pi_k(\alpha)$: probability that an agent with outside option α matches
  - shows whether the matched agents receive their desirable houses
  - larger is better

Evaluation Metrics

- Match distribution $F_k(\alpha)$:

$$F_k(\alpha) = \frac{1}{\mu} \int_0^\alpha \pi_k(\beta) f(\beta) d\beta.$$

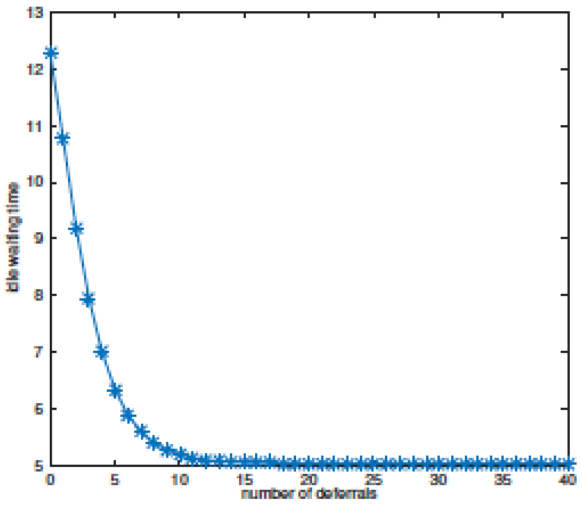
  - evaluates how many low income families benefit from the public housing
  - higher is better
- Social welfare W_k:

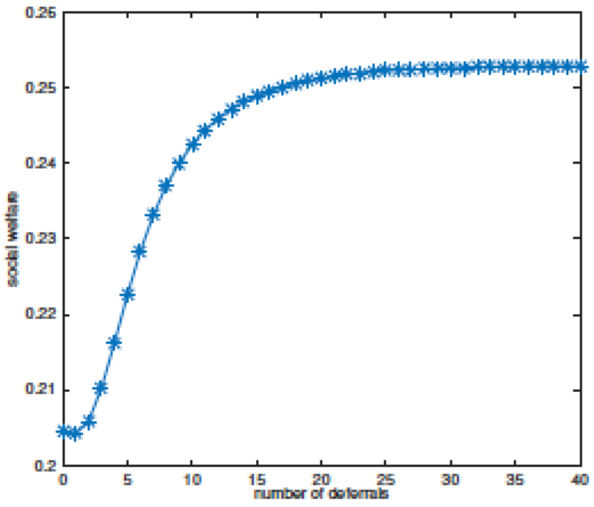
$$W_k = \frac{1}{\mu} \mathbb{E}_\alpha [u_k(\alpha)],$$

  - $u_k(\alpha) = \pi_k(\alpha)(v_k(\alpha) - \alpha)$ is the expected utility of an agent with outside option α
  - higher is better

Main Result

Under the same waiting time, we can gain 10% of the social welfare by increasing the deferral number from 2 to 5 .





Problem 3: Public Facility

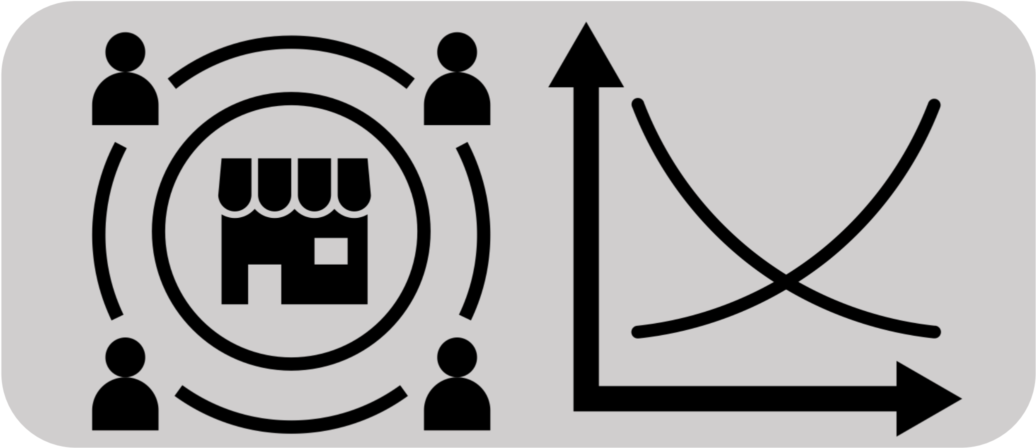
(Jalota, Pavone, Qi, Ye WINE'19)

The beginning of the story…

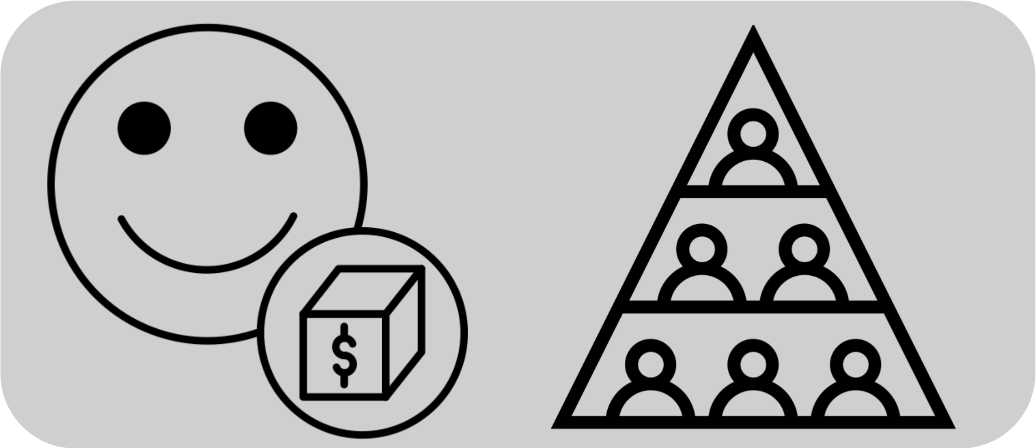
A pandemic adds **capacity restrictions** to shops, gyms, schools, and public spaces such as parks and beaches, which seem limitless under normal circumstances

We want to achieve an intermediate outcome between these contrasting scenarios

We want to develop market-based mechanisms to achieve a middle-ground between these opposing outcomes



Price Public Goods to Match Supply and Demand

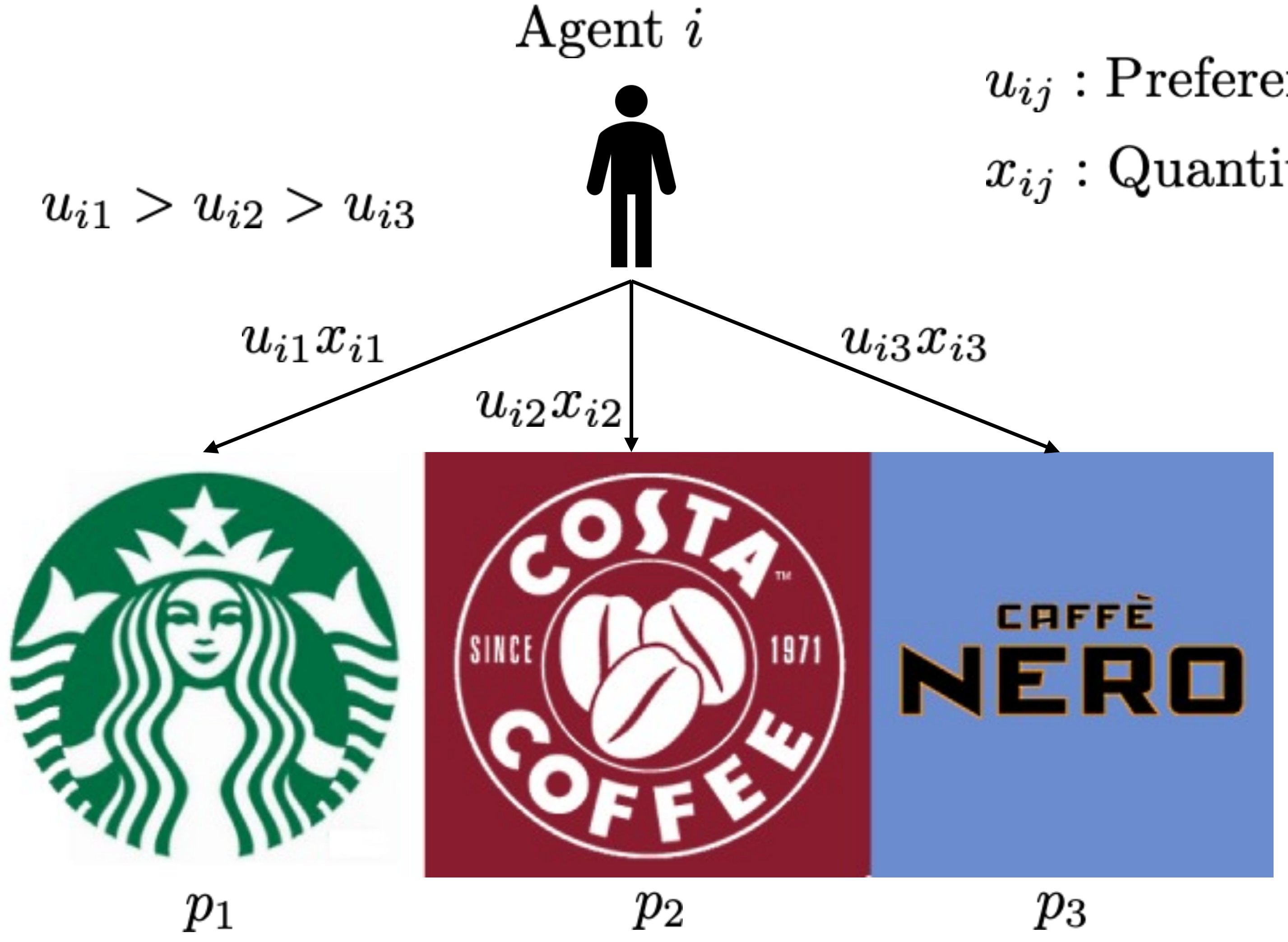


Everyone receives their most preferred bundle of goods under the set prices

The classical fisher market model

- Set budget constraints for agents, capacity constraints for resources
- Find the equilibrium price set, under which each agent buys the optimal bundle of goods.
- Each agent’s budget is fully used and each good with positive price is sold out.

Agents maximize their utilities subject to budget constraints



M = Total Number of Goods

u_{ij} : Preference of Agent i for one unit of good j

x_{ij} : Quantity of good j purchased by person i

p_j : Price of Good j

w_i : Budget of Agent i

Individual Optimization Problem:

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

$$\mathbf{x}_i \geq \mathbf{0}$$

Classical Fisher Markets provide a framework to derive prices through a centralized optimization problem

Social Optimization Problem:

$$\max_{\mathbf{x}_i, \forall i \in [N]} \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right)$$ ← Budget Weighted Log Utility

$$\text{s.t.} \quad \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$ ← Capacity Constraints

$$x_{ij} \geq 0, \forall i, j$$

p_j : Price of Good j = Dual Variable of Constraint j

u_{ij} : Preference of Agent i for one unit of good j

x_{ij} : Quantity of good j purchased by person i

p_j : Price of Good j

w_i : Budget of Agent i

$\bar{s}_j$: Capacity of Good j

The optimality conditions of the social and individual optimization problems are equivalent

Individual Optimization Problem:

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

$$\mathbf{x}_i \geq \mathbf{0}$$

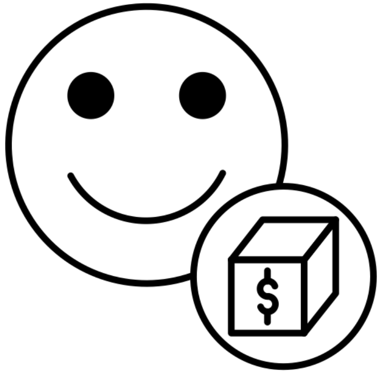
$$\Longleftrightarrow$$

Social Optimization Problem:

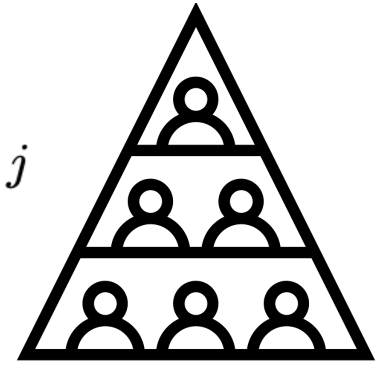
$$\max_{\mathbf{x}_i, \forall i \in [N]} \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right)$$

$$\text{s.t. } \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

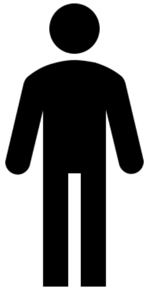
$$x_{ij} \geq 0, \forall i, j$$



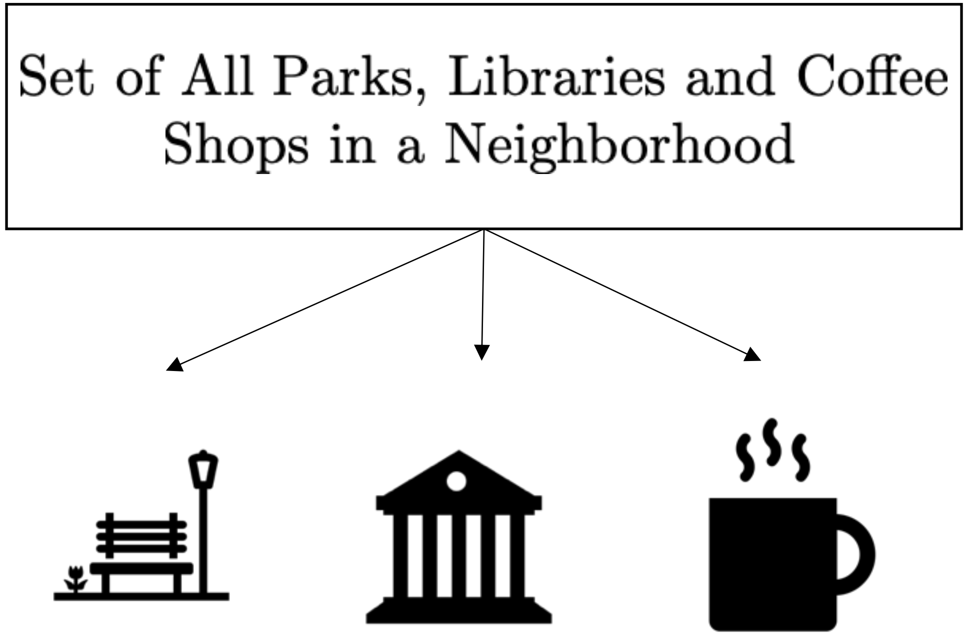
p_j : Price of Good j = Dual Variable of Constraint j



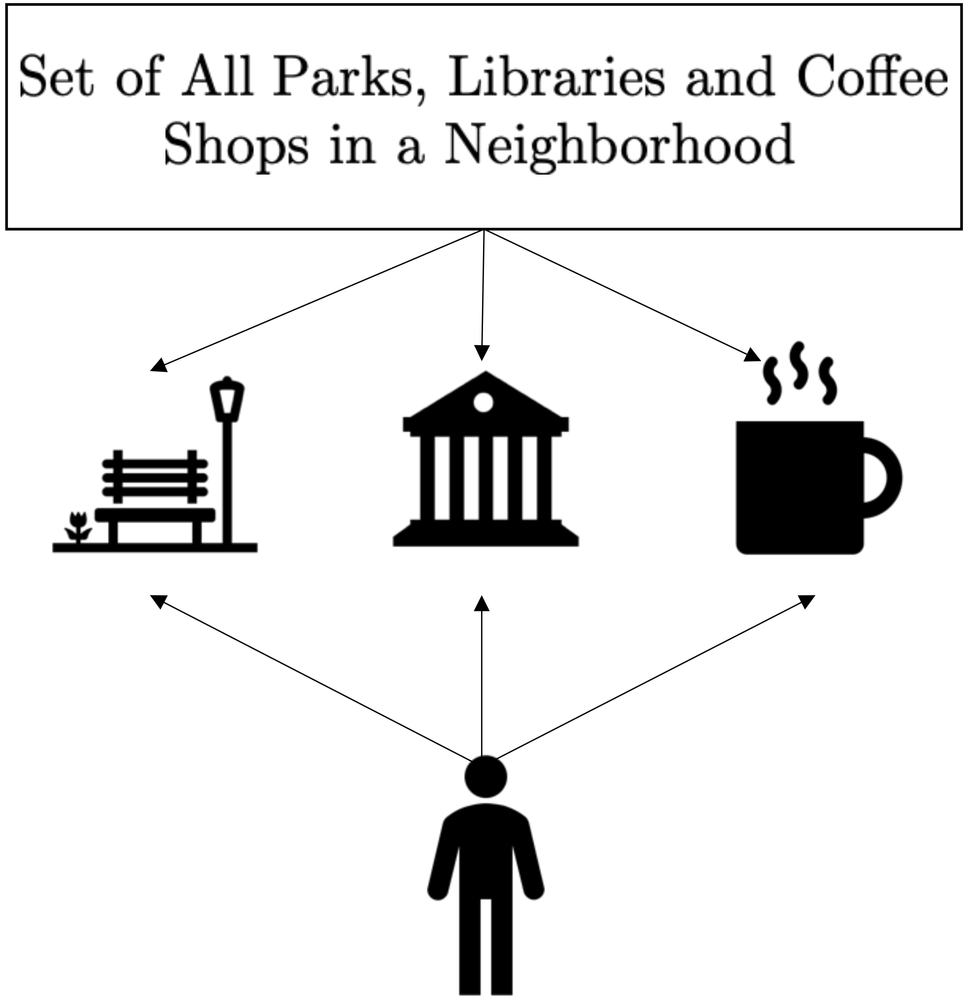
However, Fisher markets do not account for additional constraints, e.g., **knapsack constraints**, that commonly arise in public goods allocation problems



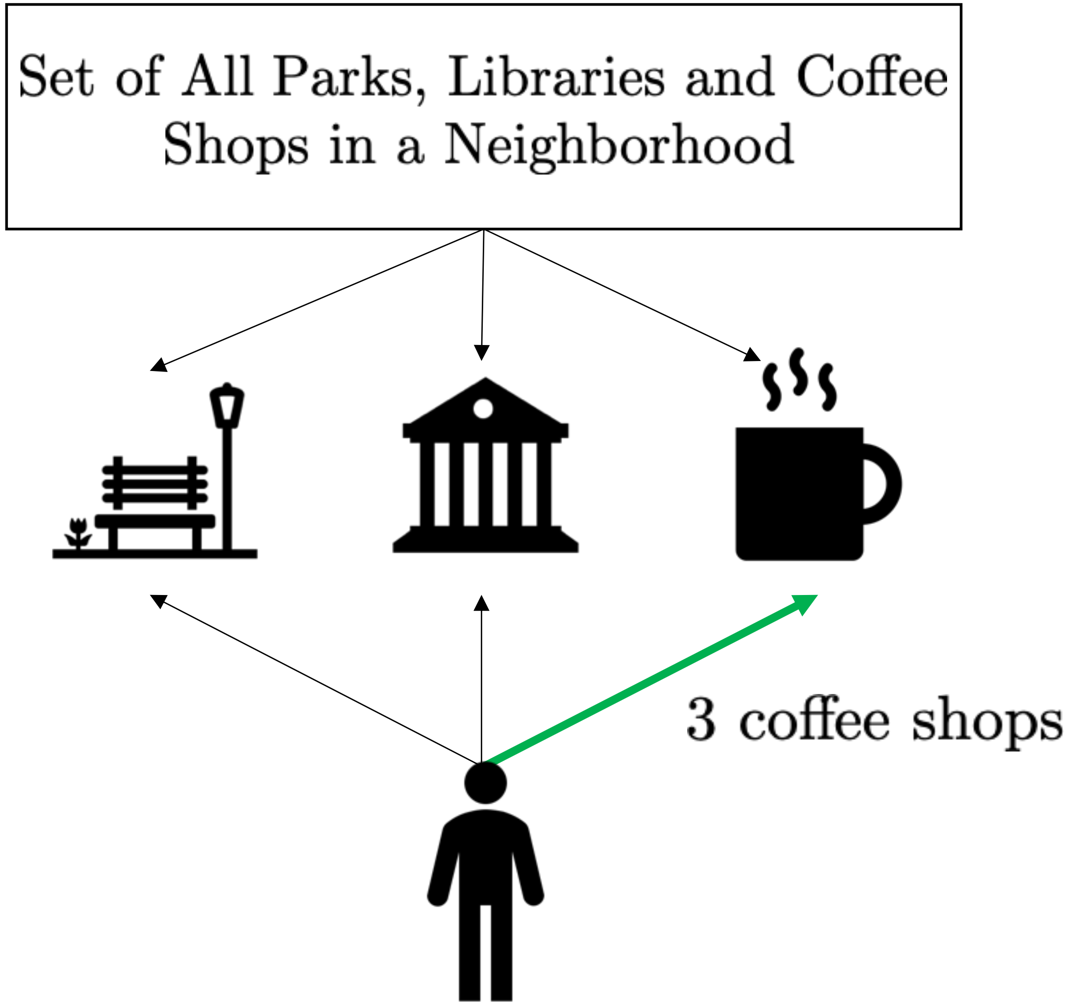
On a given day, I would like to go to one park, one library and one coffee shop



A potential allocation in Fisher markets may be…



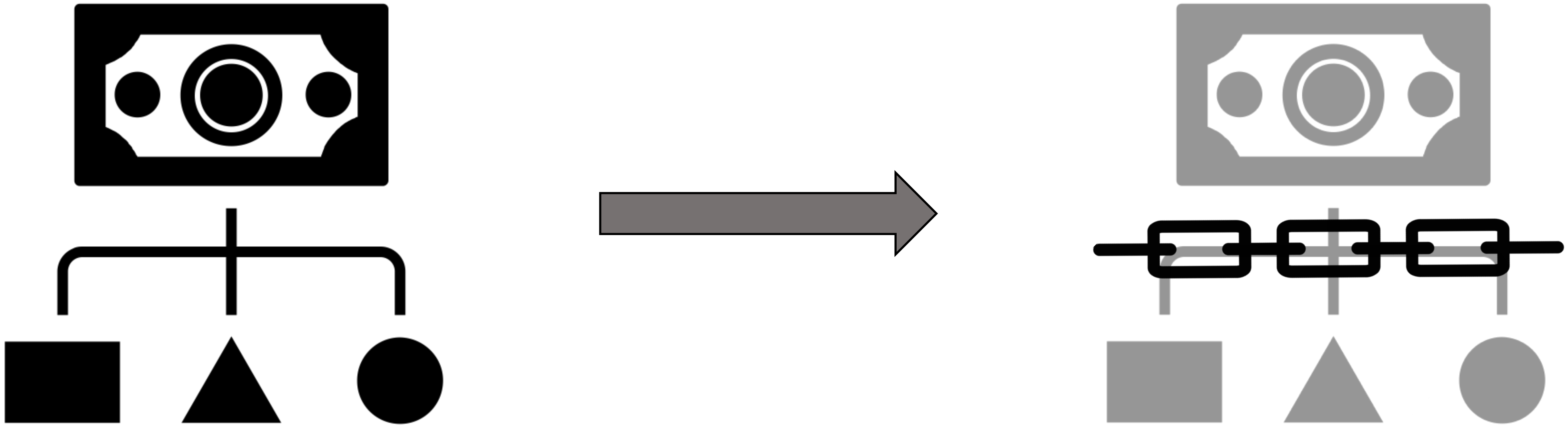
A potential allocation in Fisher markets may be…



Fisher fails

What’s next?

We extend the Fisher market framework to account for additional linear constraints



Resource Allocation under budget and capacity constraints

Resource Allocation under budget, capacity **and (e.g., knapsack)** constraints

Existing generalizations of Fisher markets are limited to spending constraints of buyers and earning constraints of sellers

Application

Supply side

- Purchasing limits
- E.g., House, car plate

- Demand side
  - Course selection, covid-19 public resource allocation

	2063	Media Technology *** view multiple offerings	Fall and Summer
☐	2331L	Photojournalism I Laboratory	Fall
☑	2332	Photo Journalism I	Fall
☐	2453	Introduction to Sports Television Production I	Fall
☐	3013	Editing	Fall and Spring

Fisher markets with additional linear constraints have different properties from classical Fisher markets

Individual Optimization Problem:

IOP

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i$$

$$\mathbf{x}_i \geq \mathbf{0}$$

Constraint Matrix

1. Market Equilibrium may not Exist

2. Market Equilibrium may not be Unique

3. People do not purchase goods with maximum *bang-per-buck*

4. May exist Giffen good

We show that under mild conditions the equilibrium exists and characterize the optimal solution of **IOP**

Individual Optimization Problem:

IOP

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t.} \quad \mathbf{p}^T \mathbf{x}_i \le w_i$$

$$A_t^{(i)} \mathbf{x}_i \le b_{it}, \forall t \in T_i$$

$$\mathbf{x}_i \ge \mathbf{0}$$

Constraint Matrix

Physical Constraints

Theorem 1. People purchase goods in the descending order of their *virtual products' bang-per-buck* ratios

Theorem 2. Market Equilibrium may not be Unique, and even not convex!

Definition 3. *(Virtual Product). A virtual product is characterized by its two endpoints* $A = (u_{ij_1}, p_{j_1})$ *and* $B = (u_{ij_2}, p_{j_2})$ *with a slope* $\theta_{j_1 j_2} = \frac{p_{j_2} - p_{j_1}}{u_{ij_2} - u_{ij_1}}$. *Then its bang-per-buck* $= \frac{1}{\theta_{j_1 j_2}} = \frac{u_{ij_2} - u_{ij_1}}{p_{j_2} - p_{j_1}}$.

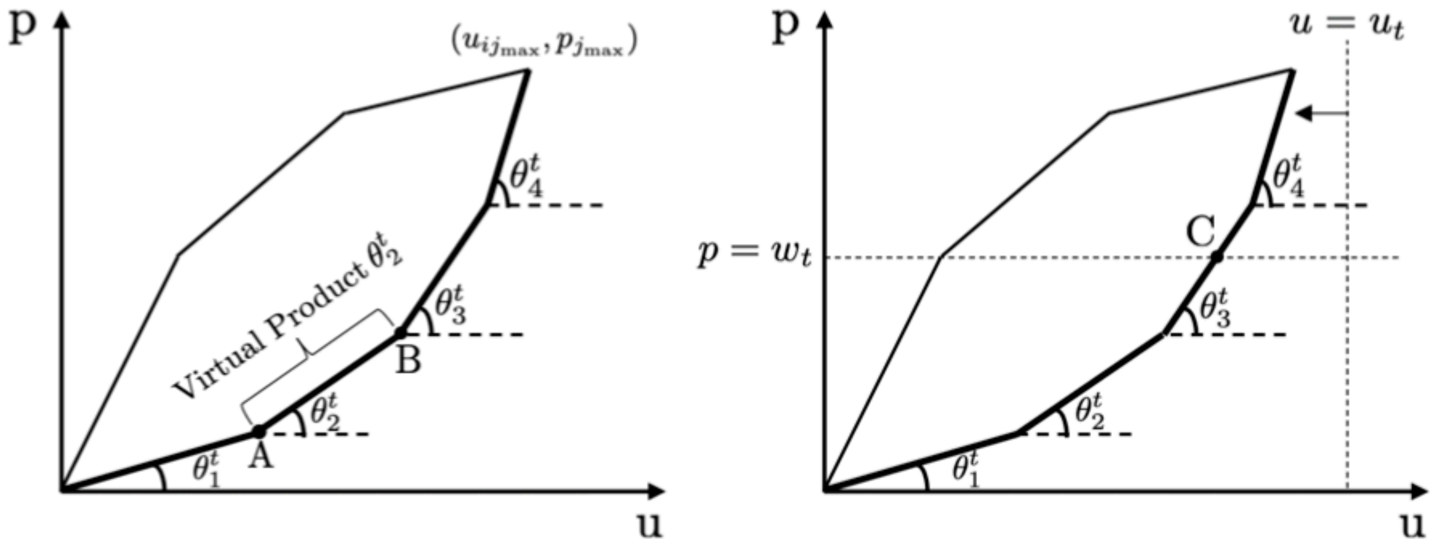


Fig. 2. The enclosed region represents the convex hull corresponding to the solution set S_t. The vertices on the lower frontier (in bold) correspond to the goods and the segments correspond to *virtual products*. The figure on the right shows that any optimal solution must lie on the lower frontier of the convex hull, as indicated by the point C.

Theorem 2. *Given a price vector* $\mathbf{p} \in \mathbb{R}^m_{\geq 0}$*, agent* i *can obtain an optimal solution* $\mathbf{x}_i^* \in \mathbb{R}^m$ *of* ***IOP*** *by mixing all virtual products from different types and spending their budget in the descending order of virtual products' bang-per-buck. Furthermore, each agent* i *can purchase at most one unit of each virtual product.*

Note that when two *virtual products* have the same slope ties can be broken arbitrarily. Further, an immediate corollary follows, which answers the question of how many different goods an agent will purchase in each *type*.

Corollary 1. *For any agent* i*, there exists an optimal solution* $\mathbf{x}_i^* \in \mathbb{R}^m$*, such that* i *purchases two different goods in at most one resource type. For all other resource types, agent* i *buys at most one good.*

We gave an example where the equilibrium price set is a curve in the 3-dimension space, which is not convex intuitively.

Proof. We consider a market with 4 agents and 4 products, with one knapsack constraint for goods 1-3 (and so we have a constraint $x_{i1} + x_{i2} + x_{i3} \leq 1$ for all i). Furthermore, since good 4 is not tied to any additional linear constraints, agents can purchase any amount of good 4 that is affordable. Next we consider the utility and budget values for each of the agents as well as the price and capacities for each of the goods as specified in Table A.4.

Utility	**Good 1**	**Good 2**	**Good 3**	**Good 4**	**Budget**
Buyer 1	2	0.0001	4	0.0001	2
Buyer 2	1	2	0.0001	0.0001	1.5
Buyer 3	0.0001	3	4	0.0001	2.5
Buyer 4	0.0001	0.0001	0.0001	1	1
Supply	1	1	1	1	
Price	$p_1 = 2 + \frac{2\eta-1}{12\eta^2+1}$	$p_2 = 2 - \frac{4\eta}{12\eta^2+1}$	$p_3 = 2 + \frac{2\eta+1}{12\eta^2+1}$	$p_4 = 1$	

Table A.4: Utilities and budgets of buyers as well as prices and capacities of goods in a four buyer, four good market to establish that the equilibrium price set may be non-convex.

We will now show that for all $\eta \in [-\frac{1}{24}, 0]$ that (p_1, p_2, p_3, p_4) is an equilibrium price set to establish non-uniqueness and then establish this set is non-convex.

Can the Fisher Market social optimization problem with additional constraints be used to set equilibrium prices?

Individual Optimization Problem:

IOP

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i$$

$$\mathbf{x}_i \geq \mathbf{0}$$

Constraint Matrix

$$? \atop \Longleftrightarrow$$

Social Optimization Problem:

SOP-I

$$\max_{\mathbf{x}_i, \forall i \in [N]} \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right)$$

$$\text{s.t. } \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i, \forall i \in [N]$$

$$x_{ij} \geq 0, \forall i, j$$

Theorem 4: The dual variables of the capacity constraint of **SOP-I** may not be an equilibrium price.

Individual Optimization Problem:

IOP

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \leq w_i$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i$$

$$\mathbf{x}_i \geq \mathbf{0}$$

Constraint Matrix

$\not\Leftrightarrow$

Social Optimization Problem:

SOP-I

$$\max_{\mathbf{x}_i, \forall i \in [N]} \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right)$$

$$\text{s.t. } \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i, \forall i \in [N]$$

$$x_{ij} \geq 0, \forall i, j$$

Remark: It is an equilibrium if the constraints are homogeneous.

A robust approach to account for physical constraints in Fisher Markets can be achieved through Budget Perturbations

SOP-I

$$\max_{\mathbf{x}_i, \forall i \in [N]} \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right)$$

$$\text{s.t.} \quad \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i, \forall i \in [N]$$

$$x_{ij} \geq 0, \forall i, j$$

A robust approach to account for physical constraints in Fisher Markets can be achieved through Budget Perturbations

SOP-I

$$\max_{\mathbf{x}_i, \forall i \in [N]} \sum_i w_i \log \left(\sum_j u_{ij} x_{ij} \right)$$

$$\text{s.t.} \quad \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i, \forall i \in [N]$$

$$x_{ij} \geq 0, \forall i, j$$

BP-SOP

Budget Perturbation

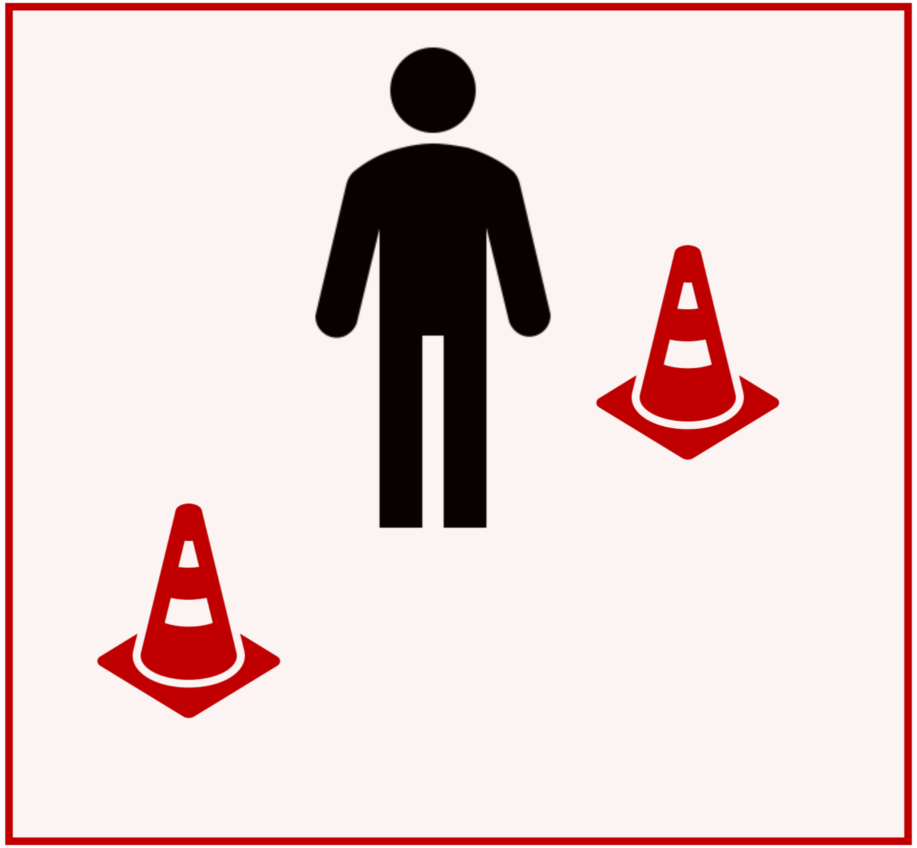
$$\max_{\mathbf{x}_i} \sum_i (w_i + \lambda_i) \log \left(\sum_j u_{ij} x_{ij} \right)$$

$$\text{s.t.} \quad \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

$$A_t^{(i)} \mathbf{x}_i \leq b_{it}, \forall t \in T_i, \forall i \in [N]$$

$$x_{ij} \geq 0, \forall i, j$$

Budget Perturbations allow more constrained agents, e.g., medical workers during a pandemic to have “higher priority” to get their goods



Low λ_i

High λ_i

X

Theorem 5: The dual variables of the capacity constraint of **BP-SOP** are the market equilibrium price iff $\lambda_i = \sum_t r_{it} b_{it}$

Individual Optimization Problem:

IOP

$$\max_{\mathbf{x}_i} \sum_j u_{ij} x_{ij}$$

$$\text{s.t. } \mathbf{p}^T \mathbf{x}_i \le w_i$$

$$A_t^{(i)} \mathbf{x}_i \le b_{it}, \forall t \in T_i$$

$$\mathbf{x}_i \ge \mathbf{0}$$

$$\Longleftrightarrow$$

Social Optimization Problem:

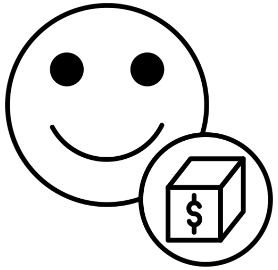
BP-SOP

$$\max_{\mathbf{x}_i} \sum_i (w_i + \lambda_i) \log \left(\sum_j u_{ij} x_{ij} \right)$$

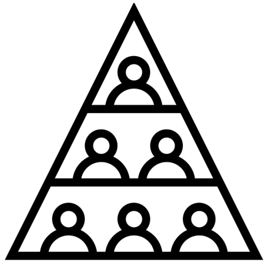
$$\text{s.t. } \sum_i x_{ij} = \bar{s}_j, \forall j \in [M]$$

r_{it}: Dual Variable $\quad A_t^{(i)} \mathbf{x}_i \le b_{it}, \forall t \in T_i, \forall i \in [N]$

$$x_{ij} \ge 0, \forall i, j$$



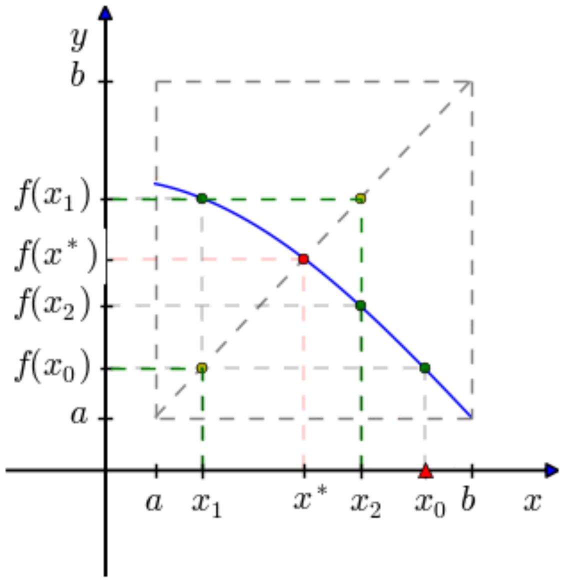
p_j : Price of Good j = Dual Variable of Constraint j

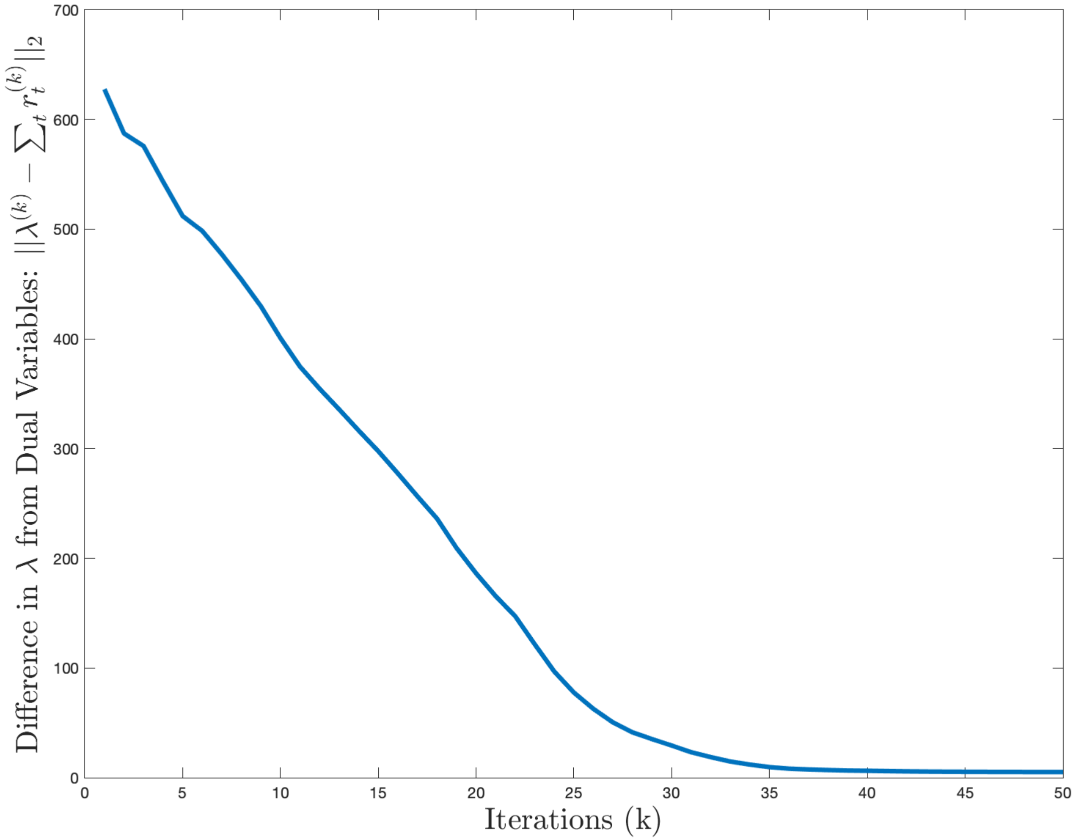


To determine the perturbation constants we use a fixed-point iterative procedure

Algorithm 1: Fixed Point Scheme

Input : Tolerance ϵ, Function $G(\cdot)$ to calculate dual variables
Output: Budget Perturbation Parameters λ
$\lambda \leftarrow \mathbf{0}$;
$\mathbf{R} \leftarrow G(\lambda)$;
$q_i \leftarrow \sum_{t=1}^{l_i} r_{it} b_{it}, \forall i$;
while $\|\lambda - \mathbf{q}\|_2 > \epsilon$ **do**
  $\lambda_i \leftarrow \sum_{t=1}^{l_i} r_{it} b_{it} \ \forall i$;
  $\mathbf{R} \leftarrow G(\lambda)$;
  $q_i \leftarrow \sum_{t=1}^{l_i} r_{it} b_{it}, \forall i$;
end





Methods to compute market equilibrium

Convex optimization

Tatonnement process

Primal-dual

Auction based approach

…

Centralized vs. distributed

Alternating Direction Methods (ADMs)

A new class of distributed tatonnement algorithms

Alternating Minimization Algorithm (AMA)

- Agents distributed solve their individual optimization problems at each iteration
- The step sizes of the price updates based on utilities

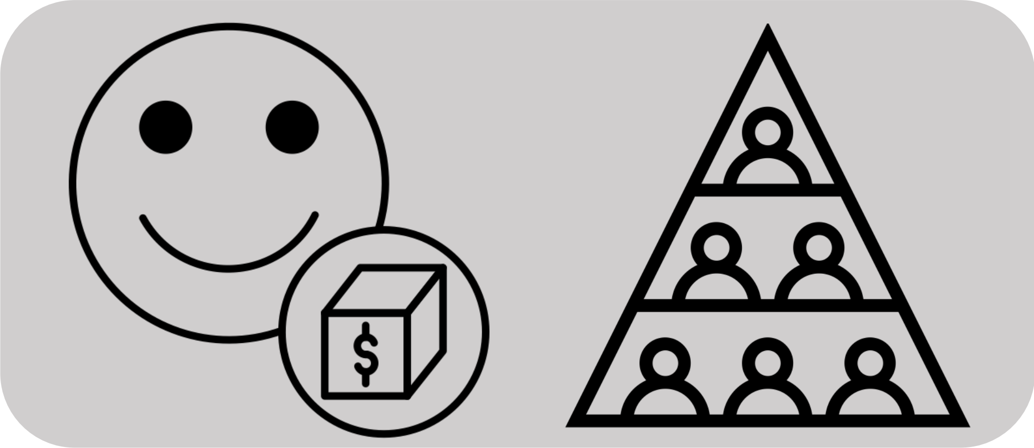
Alternating Direction Method of Multipliers (ADMM) Algorithm

- The step size is independent of utilities

Both algorithms O(1/k) convergence rate for fisher market with homogeneous linear constraints

We design a market mechanism that derives market clearing prices while supporting additional constraints

Set Prices that Clear the Market, i.e., all goods are sold, and all budgets are used



Maximize a Social Objective while each person obtains their most preferred bundle of goods

Thank you!