Fair Division with Money and Prices

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Summer Scholl on Game Theory and Social Choice

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June 20, 2022

joint work with Anna Bogomolnaia

practical mechanism design: simple rules to choose a public decision or divide private goods

Voting by Veto, Divide & Choose, Final Offer Arbitration, Bundled Auction are attractive alternatives to direct bargaining:

avoid costly haggling: minimal interactions between participants
simple messages reveal little of own preferences, require little cognitive effort

safe play (maximising one's worst case utility) delivers high individual guarantees
even to agents clueless about each other's preferences

the downside: simple input messages $\Longrightarrow$ the outcome of the mechanism is *inefficient*

a challenging question: what are the trade offs between

$\longrightarrow$ the need for simple messages

$\longrightarrow$ the loss of efficiency under decentralised safe play

our problem: two agents U and V are the joint owners of a set of indivisible goods $A = \{a, b, \cdots\}$ (disposable objects)
they must share the goods, using cash transfers to "smooth out" the indivisibilities

utilities are linear over money: $(S, t) \rightarrow u(S) + t$, where $u(\varnothing) = 0$

$$\text{full domain: } \mathcal{F} = \{u | 2^A \ni S \rightarrow u(S) \in \mathbb{R}_+, \text{ inclusion monotonic}\}$$

the efficient allocation of the objects collects the surplus

$$\mathcal{S}(u, v) = \max_{\varnothing \subseteq S \subseteq A} \{u(S) + v(A \diagdown S)\}$$

Examples: dissolution of partnership, division of the rent between flat-mates, assigning noxious or desirable facilities to communities, workloads between partners etc..

well studied in the **assignment** case: Cramton et al. (1987), Alkan (1989), Alkan Demange and Gale (1991), Tadenuma and Thomson (1991), Aragones (1995)

much less so when the allocation of objects is arbitrary: Moulin (1992) for divisible Arrow Debreu commodities; fair outcome of auctions Papai (2003), Long (2018)

the vast literature on **moneyless** allocation of indivisible objects (typically under additive utilities) is not relevant here fix the division rule (a complete mechanism) and the domain $\mathcal{D}$ of utilities

for each $u \in \mathcal{D}$ let $\Gamma(u)$ be agent U's *guarantee* offered by the rule if U plays *safely*: maximizes her utility against a fully adversarial agent V

the *minimal* efficiency performance of our rule in $\mathcal{D}$:

$$\theta(\mathcal{D}) = \min_{u,v \in \mathcal{D}} \frac{\Gamma(u) + \Gamma(v)}{\mathcal{S}(u,v)}$$

it applies to any of the strategic equilibria of the rule

or if it is used as the disagreement option in direct bargaining

let $\varphi_U(u, v)$ be the total utility collected when
U and V's messages are fully decentralised and both agents play safely

the *decentralised* efficiency performance of the rule in $\mathcal{D}$:

$$\sigma(\mathcal{D}) = \min_{u,v \in \mathcal{D}} \frac{\varphi_U(u, v) + \varphi_V(u, v)}{\mathcal{S}(u, v)}$$

$$\theta(\mathcal{D}) \leq \sigma(\mathcal{D})$$

the **Bundled Auction** (BA) (Texas Shoot Out)
each agent bids for A: the winner (highest bidder) gets A and pays their bid to the loser

the safe bid is $x = \frac{1}{2}u(A)$ which guarantees the *Fair Share* $\Gamma^{FS}(u) = \frac{1}{2}u(A)$

$$\theta^{BA}(\mathcal{F}) = \sigma^{BA}(\mathcal{F}) = \frac{1}{2}$$

but the BA rule *(played safely or not)* cannot split the objects when it is efficient to do so
A is made of identical marbles and the marginal utility of one more marble decreases
or the marbles are of two colors, U wants only blue ones and V only red ones
or U can only use marbles of the same color, whatever the color
etc..

our new Sell&Buy division rule

defined first in the sub-domain $\mathcal{ID}$: goods are identical: $u(S) = u(|S|)$

example: 30 bags of sweets, day-passes to a park, hours of baby sitting

utilities can be additive (resale value only) subadditive (storage costs) super-additive (paying a bill)

step 1 agent U bids x and agent V bids y: to become the Seller

if $x < y$ the Seller is U (no money changes hands in this step); tie-breaking is arbitrary

step 2 the Buyer (agent V) buys from the Seller any number of goods at unit price $\frac{x}{m}$; say he buys k goods, $0 \leq k \leq m$, the final allocation is

$$U : (m - k \text{ goods}, +\frac{k}{m}x),\ V : (k \text{ goods}, -\frac{k}{m}x)$$

contrast the maximally *subadditive* Frugal agent u^F: one good is enough
and the maximally *superadditive* Greedy u^G: give me all or nothing

	$\varnothing$	$\varnothing \subsetneq S \subsetneq A$	A
Greedy: u^G	0	0	$u^G(A)$
Frugal: u^F	0	$u^F(A)$	$u^F(A)$

Frugal bids $\frac{m}{m+1}u^F(A)$ and is guaranteed $\Gamma(u^F) = \frac{m}{m+1}u^F(A)$
Greedy bids $\frac{m}{m+1}u^G(A)$ and is guaranteed $\Gamma(u^G) = \frac{1}{m+1}u^G(A)$

rewarding/penalising sub/super additive is normatively appealing *how efficient is safe play in the $\mathcal{ID}$ domain ?*

fully efficient under S&B and BA in the problems (Frugal,Greedy) or (Greedy,Greedy)

in a problem (Frugal, Frugal) BA only collects $u(A) \vee v(A)$ instead of $u(A) + v(A)$ whereas S&B is fully efficient

BA can be more efficient than S&B: $u(k) = \frac{2k}{m} \vee (\frac{8k}{m} - 4)$ so $x^* = 2$; V is Greedy with $v(m) = 3$ so $y^* = \frac{3m}{m+1} > 2$ and V buys the whole A
the S&B rule collects 3 while the efficient surplus is 4

Proposition 1: *if goods are identical*

$$\frac{1}{m+1} = \theta^{SB}(\mathcal{ID}) < \sigma^{SB}(\mathcal{ID}) = \frac{1}{2}\frac{m+2}{m+1}$$

if u *is subadditive** $(u(A) \le u(S) + u(A \setminus S)$ *for all* $S)$ *then* $\Gamma(u) \ge \frac{1}{2}u(A)$

compare to $\theta^{BA}(\mathcal{ID}) = \sigma^{BA}(\mathcal{ID}) = \frac{1}{2}$
but numerical experiments show a significantly higher **average** efficiency for S&B

define S&B in the sub-domain $\mathcal{ADD}$: additive utilities for heterogenous objects

first step as before

step 2 the Seller U posts a price vector $p \in \Delta(x)$: $p \geq 0$ $\sum_{a \in A} p_a \overset{\text{notation}}{=} p_A = x$

step 3 the Buyer V chooses a share S of objects ($\varnothing \subseteq S \subseteq A$) and pays p_S to the Seller

final allocation

$$\text{Buyer } V\text{: } (S, -p_S) \text{ , Seller } U \text{ : } (A \diagdown S, +p_S)$$

Lemma: *for any* $u = (u_a)_{a \in A} \in \mathcal{ADD}$ *the safe bid in S&B is* $x = \frac{1}{2}u(A)$ *followed as Seller by posting* $p = \frac{1}{2}u$*; this guarantees* $\Gamma^{SB}(u) = \frac{1}{2}u_A$

example where BA is obviously inefficient and S&B is efficient

	a	b	c	d
u	3	5	1	2
v	1	2	5	6

safe play by both agents in S&B captures the full surplus of 19
while in BA it gets only 14

	a	b	c	d
u	3	5	3	3
v	1	2	5	5

now both S&B and BA capture 14 out of the efficient 18

Proposition 2: *if utilities are additive*

$$\frac{1}{2} = \theta^{SB}(\mathcal{ADD}) < \sigma^{SB}(\mathcal{ADD}) = \frac{2}{3}$$

compare to $\theta^{BA}(\mathcal{ADD}) = \sigma^{BA}(\mathcal{ADD}) = \frac{1}{2}$

Note: in $\mathcal{ADD}$ the *multi-Auction* rule is efficient and fair, but it does not work at all outside this domain: example Greedy or $u(k) = \mathbf{1}_{\{k \geq \frac{m+1}{2}\}} \cdot u(A)$

the full domain $\mathcal{F}$

$$\varnothing \subseteq S \subset T \subseteq A \Longrightarrow u(\varnothing) = 0 \leq u(S) \leq u(T)$$

of exponential complexity in $|A|$

if $|A| \geq 5$ human agents will not report a full utility
but *simple* messages such as a price vector on the goods

cognitively feasible (e. g. in Spliddit)
keeps utilities *mostly* private

the definition of S&B in the full domain $\mathcal{F}$ is the same as in $\mathcal{ADD}$

how to play safe in the S&B rule?

if you bid too high you risk becoming the Buyer facing a high price
if you bid too low you risk being cheaply sold out

compare my worst utility $W(u;x)$ if I am the Seller (my bid x wins: $x \leq y$)
and my worst utility $L(u;x)$ if I am the Buyer ($x > y$ and y is just below x)

$$W(u;x) = \max_{p \in \Delta(x)} \min_{\varnothing \subseteq T \subseteq A} (u(T) + p_{A \setminus T}) = \max_{p \in \Delta(x)} \{ \min_{\varnothing \subseteq S \subseteq A} (u(S) - p_S) + x \}$$

$$L(u;x) = \min_{p \in \Delta(x)} \max_{\varnothing \subseteq T \subseteq A} (u(T) - p_T)$$

Theorem:

$i)$ $x \longrightarrow W(u;x)$ *is concave and strictly increasing from* 0 *to* $u(A)$
$x \longrightarrow L(u;x)$ *is convex and strictly decreasing from* $u(A)$ to 0
they intersect uniquely at x^* and the corresponding guarantee is

$$\Gamma^{SB}(u) = W(u;x^*) = L(u;x^*)$$

$ii)$ *this guarantee is maximal (unimprovable)*

$iii)$ *both mappings* $u \longrightarrow x^*$ *and* $u \longrightarrow \Gamma^{SB}(u)$ *are*
continuous and weakly increasing in u
scale invariant and weakly increasing in A

$iv)$ *the following bounds are exact*

$$\frac{1}{2}\partial_a(u)^{\mathsf{min}} \leq p_a \leq \partial_a(u)^{\mathsf{max}}$$

where $\partial_a(u)^{\mathsf{max}} = \mathsf{min}_{\varnothing \subseteq S \subseteq A}\, \partial_a u(S)$; $\partial_a(u)^{\mathsf{max}} = \mathsf{max}_{\varnothing \subseteq T \subseteq A}\, \partial_a u(T)$

$$\frac{1}{2}u(A) \leq x^* \leq \frac{\sqrt{m}}{2}u(A)$$

note: the inequality $x^* \leq u(A)$ is often true, e. g. if the goods are identical, or additive, or u is balanced

v) *the following bounds are exact*

$$\frac{1}{m+1}u(A) \leq \Gamma^{SB}(u) \leq \frac{m}{m+1}u(A)$$

$$\Gamma^{SB}(u) \geq \frac{1}{2}u(A) \text{ if } u \text{ is sub-additive}^*$$

$$\Gamma^{SB}(u) \leq \frac{1}{2}u(A) \text{ if } u \text{ is super-additive}^*$$

Proposition 3: $i)$

$$\frac{1}{m+1} \leq \theta^{SB}(\mathcal{F}) \leq \sigma^{SB}(\mathcal{F}) \leq \frac{2}{\sqrt{m}}$$

we conjecture that $\sigma^{SB}(\mathcal{F}) = O(\frac{1}{\sqrt{m}})$

$ii)$ *over the sub-additive* domain* $\mathcal{SUB}$ $(u(A) \leq u(S) + u(A \diagdown S)$ for all $S)$

$$\theta^{SB}(\mathcal{SUB}) = \sigma^{SB}(\mathcal{SUB}) = \frac{1}{2}$$

$ii)$ *over the super-additive* domain* $\mathcal{SUB}$ $(u(A) \geq u(S) + u(A \diagdown S)$ for all $S)$

$$\frac{2}{m+1} = \theta^{SB}(\mathcal{SUB}) < \sigma^{SB}(\mathcal{SUB}) < \frac{1}{2}$$

two sufficient conditions ensuring that the safe play in S&B is efficient

Proposition 4 *suppose agent U dominates agent V utility-wise*

$$\partial_a(u)^{\mathsf{min}} > \partial_a(v)^{\mathsf{max}} \text{ for all } a \in A$$

then the S&B rule (just like BA) safely implements the efficient surplus by giving all goods to U

Proposition 5: *suppose the goods are split into two piles B, C and*

$$\partial_b(u)^{\mathsf{min}} > 2 \times \partial_b(v)^{\mathsf{max}} \text{ in } B \text{ and } \partial_c(v)^{\mathsf{min}} > 2 \times \partial_c(u)^{\mathsf{max}} \text{ in } C$$

then the S&B rule (unlike BA) safely implements the efficient surplus by giving B to U and C to V

the Seller-Buyer division rule and its guarantee generalise to problems

- with any number of agents

- with bad objects, mimicking the results for problems with good objects; or with an *objective* mixture of goods and bads

- where agents have unequal rights to (or liabilities from) the surplus (or losses) from the objects

- the S&B rule elicit a full price vector from only one of the two agents; it is cognitively feasible and preserves the privacy of full utilities

- the utility guaranteed by S&B rewards sub-additive* utilities and penalises super-additive* ones
  together the two guarantees reduce substantially the bargaining gap: difference in surplus between the worst and the best allocations

- safe play is the purely decentralised form of strategic behaviour: in S&B it captures *on average* a substantial fraction of the efficient surplus

systematic numerical experiments to confirm the last two statements are on-going

THANK YOU

an example with complementary goods: A contains a-goods and b-goods, 8 of each type

$$u(\alpha, \beta) = \min\{\lfloor \frac{\alpha}{3} \rfloor, \beta\} \; ; \; v(\alpha, \beta) = \min\{\alpha, \{\lfloor \frac{\beta}{3} \rfloor\}$$

under the S&B rule each agent bids $x = y = \frac{56}{33} \simeq 1.70$

if U is chosen as Seller he posts the price $p_a = \frac{4}{33} \simeq 0.12$, $p_b = \frac{1}{11} \simeq 0.10$ then the Buyer V purchases 6 b-goods and 2 a-goods

this results in the efficient allocation for a surplus of 4
while in BA the surplus collected is only 2

Proof of statement $i)$ in Proposition 3

assume $m = \ell^2$ and arrange the goods in a $\ell \times \ell$ matrix $A = [a_{ij}]_{1 \leq i,j \leq \ell}$

$u(S) = 1$ if S contains the top row $\{a_{11}, a_{12}, \cdots, a_{1\ell}\}$ of A, else $u(S) = 0$
$\Longrightarrow U$ bids $x^* = \frac{\ell}{\ell+1}$

$v(S) = \frac{2+\varepsilon}{\ell+1}$ if S contains a (any) column of A, else $v(S) = 0$
$\Longrightarrow V$ bids $y^* = (\frac{2^+}{\ell+1}) \times \frac{\ell}{2}$

then the Seller U safely asks $\frac{1}{\ell+1}$ for each good in the top row (others are free) and V buys any column for just $\frac{1}{\ell+1}$: the surplus collected is $\frac{2^+}{\ell+1} \simeq \frac{2}{\sqrt{m}}$

two normative requirements on a guarantee, inspired by the Frugal/Greedy example

$$\text{Positive: } u \neq 0 \Longrightarrow \Gamma_n(u) > 0$$

$$\text{Responsive: } \Gamma_n(u^G) < \Gamma_n(u^F)$$

both OK for S&B but the FS is not responsive *two interpretations of ex post fairness*: the most familiar one

EnvyFreeness(EF): $u_i(S_i) + t_i \geq u_i(S_j) + t_j$

and the alternative

No Subsidy (NS): $u_i(S_i) + t_i \leq u_i(A)$

I wish EF for myself and NS for you

Envy Free and No Subsidy are incompatible safe play and ex post efficiency

playing safe in S&B will someti(or in D&C) will sometimes violate No Subsidy

Lemma: *Envy cannot be avoided if the rule implements a Positive and Responsive guarantee*

Proof in a two-person problem Greedy vs Frugal
there are two types of envy free allocations
someone eats A and pays $\frac{1}{2}$ to the other; or they each get at least one object and no transfer takes place
the former is not Responsive, the latter is not Positive

Proposition 6 *in the S&B rule the Buyer is never subsidised; neither is the (safe) Winner if his bid is at most* $u(A)$*, which happens*

- *in any problem with at most four objects*
- *if the objects are identical*
- *if each utility is either balanced or anti-balanced (e. g. sub- or super-modular)*