Fair Division of Indivisible Goods

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Discrete Fair Division

Given

- Set $[n]$ of n agents, $i, j \ldots$ are agents.
- Set M of m indivisible goods. house, car, toothbrush, ...
- Valuation of agent i: $v_i : 2^M \to \mathbb{R}_{\geq 0}$.

  $v_i(S) =$ value of bundle S to agent i,
  $v_i(\emptyset) = 0$ and $v_i(A) \leq v_i(B)$ if $A \subseteq B$.
- Valuation is additive if $v_i(S) = \sum_{g \in S} v_i(g)$ for all bundles S.
- Valuations are additive, if not said otherwise.

Find: A fair partition $X = \langle X_1, X_2, \ldots, X_n \rangle$ of M.

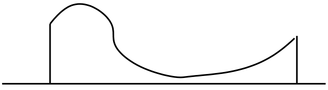
X_i is the bundle assigned to agent i.

Outline

- Fair Divison of a Cake
- Notions of Fairness: EF (envy-free), EF1, EFX, NSW (Nash Social Welfare), MinMax-Share, ...
- EF is too much to ask for, but EF1 always exists.
- EFX exists for 3 agents; existence for four+ agents is open.
- Relaxations of EFX
  - EFX with charity, i.e., some unassigned goods.
  - Nash Social Welfare
    - exists, but is hard to approximate (APX-hard).
    - Constant factor approximations.
    - Two-Valued Instances: surprisingly rich.

Talk is based on papers in EC20, EC21, JACM 22, FSTTCS 20, SODA 21, AAAI 22, SICOMP 22, arXive 22, unpublished.

Fair Division of a Cake (One Divisible Good)



Each v_i is a density function on the unit interval (= area under the curve is equal to one).

Goal: A envy-free partition $X = \langle X_1, X_2, \ldots, X_n \rangle$ of the unit interval, i.e., for any $i, j \in [n]$, $v_i(X_j) \leq v_i(X_i)$.

Problem was introduced by Steinhaus in the '40s.

For $n = 2$, there is a simple solution: Cut and Choose.

For $n > 2$, the problem is complex.

- $n = 3$, independent solutions by J. Selfridge and J. Conway in '60.
- arbitrary n, solution by S. Brams and A. Taylor, '95.
- arbitrary n, solution by H. Aziz and S. Mackenzie, '16, # of cuts bounded by function of n.

Applications of Discrete Fair Division

Ad of a Saarbrücken law firm:
Your brother enjoys your bequest in the sun.

The websites Spliddit and Fair Outcome offer algorithms for fair division problems, e.g.,

- dividing goods (divorce settlement, inheritance property),
- splitting rent,
- dividing chores (papers to review or household duties).

Quintessential Notion: Envy-Freeness

X is envy-free = iff for all pairs (i, j) we have $v_i(X_i) \geq v_i(X_j)$, i.e.,
i likes his bundle at least as much as any other bundle.

Question

Is there always an envy-free allocation?

Quintessential Notion: Envy-Freeness

X is envy-free = iff for all pairs (i, j) we have $v_i(X_i) \geq v_i(X_j)$, i.e., i likes his bundle at least as much as any other bundle.

Question

Is there always an envy-free allocation?

Answer

NO! Consider two agents having positive valuation towards a single good.

How about relaxations?

Relaxation: Envy-Freeness up to One Good (EF1) [Budish'11]

X is EF1 iff for all i,j, $v_i(X_i) \geq v_i(X_j \setminus \{ g \})$ for some $g \in X_j$, i.e., envy disappears after removing the most valuable good according to i.

Agents	House	Car	Toothbrush
a_1	10000	100	1
a_2	10000	100	1

$$X_1 \leftarrow \{ \text{House, Toothbrush} \}.$$
$$X_2 \leftarrow \{ \text{Car} \}.$$

An EF1 Allocation

Envy-Freeness upto One Good (EF1)

X is EF1 iff for all i,j, $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for **some** $g \in X_j$.

Question

Is there always an EF1-allocation?

Answer [Lipton et al.'04]

YES

Envy-Freeness upto One Good (EF1)

X is EF1 iff for all i,j, $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for **some** $g \in X_j$.

Question

Is there always an EF1-allocation?

Answer [Lipton et al.'04]

YES

But EF1 is not a satisfactory notion of fairness.

An agent may rightfully say: I have understood that some envy is unavoidable. But, if I envy you, the envy should go away after removing **any** good from your bundle. In particular, it should go away after removing the good which I value **least**.

Weaker Relaxation: Envy Freeness up to any Good (EFX) [Caragiannis/Kurokawa/Moulin/Procaccia/Shah/Wang '16]

X is EFX iff for all i,j, $v_i(X_i) \geq v_i(X_j \setminus \{ g \})$ for all $g \in X_j$, i.e., envy disappears after removal of the least value good (least valuable to i).

Agents	House	Car	Toothbrush
a_1	10000	100	1
a_2	10000	100	1

$$X_1 \leftarrow \{ \text{House, Toothbrush} \}.$$
$$X_2 \leftarrow \{ \text{Car} \}.$$

This allocation is not EFX.

Weaker Relaxation: Envy Freeness up to any Good (EFX) [Caragiannis/Kurokawa/Moulin/Procaccia/Shah/Wang '16]

X is EFX iff for all i,j, $v_i(X_i) \geq v_i(X_j \setminus \{ g \})$ for all $g \in X_j$, i.e., envy disappears after removal of the least value good (least valuable to i).

Agents	House	Car	Toothbrush
a_1	10000	100	1
a_2	10000	100	1

$$X_1 \leftarrow \{ \text{House} \}.$$
$$X_2 \leftarrow \{ \text{Car, Toothbrush} \}.$$

This allocation is EFX.

Envy Freeness up to any Good (EFX)

X is EFX iff for all i,j, $v_i(X_i) \geq v_i(X_j \setminus \{ g \})$ for all $g \in X_j$.

An agent strongly envies a set of goods if it envies a proper subset of it.

An allocation is EFX iff no agent strongly envies (the bundle of) any other agent.

Question

Is there always an EFX allocation?

Answer - We do not know yet!

Fair division's biggest problem — Ariel Procaccia [CACM'20]

EFX exists for $n = 2$ and Additive Valuations

Refinement of Cut and Choose.

Algorithm for Agent 1

Create two empty bundles;
for all goods g in decreasing order of value to agent 1 **do**
    assign g to the bundle of smaller value
    Invariant: Advantage of more valuable bundle $\leq$
        least valuable good (acc. to 1) in this bundle;
        agent 1 is happy with either bundle.
Return $\langle X_1, X_2 \rangle$.

Agent 2 picks his preferred bundle.

State of the Art (EFX)

- identical valuations; EFX exists for all n, Plaut/Roughgarden, SODA '18
- additive valuations;
  - $n = 3$, EFX exists, Chaudhury/Garg/Mehlhorn EC '20
  - $n \geq 4$, **Open**
  - any n, EFX with Charity exists, Caragiannis/Gravin/Huang '19, Chaudhury/Kavitha/M/Sgouritsa SODA '20, Chaudhury/Garg/Mehlhorn/Misra EC '21, Akrami/Chaudhury/Mehlhorn unpublished
- general valuations
  - $n = 2$, EFX exists, Plaut/Roughgarden SODA '18
  - $n \geq 3$, **Open**
  - any n, EFX with charity exists, Chaudhury/Kavitha/M/Sgouritsa

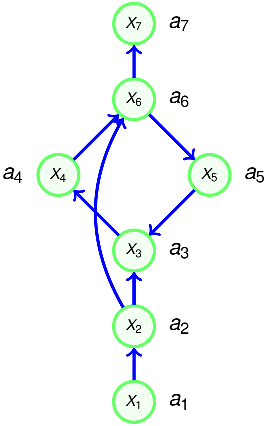
EFX with Charity (P are the goods that go to charity)

There exists a partition $\langle X_1, X_2, \ldots, X_n, P \rangle$ of M such that,

- $\langle X_1, X_2, \ldots, X_n \rangle$ is EFX,
- For every i, we have $v_i(P) \leq v_i(X_i)$ and
- $|P| < n$. and with sublinear charity ($n^{5/6}$, $n^{2/3}$)

Envy Graph G_X (Lipton et al. '04)

- Vertices correspond to agents $[n]$.
- $(i, j) \in G_X$ iff i envies j, i.e., $v_i(X_i) < v_i(X_j)$.



We can always assume that G_X is acyclic.

If there is a cycle $a_0 \to a_1 \to \ldots \to a_k = a_0$ give a_i's bundle to a_{i-1} for $1 \leq i \leq k$. In this way all agents on the cycle improve and no new envy.

Continue until the envy graph is acyclic.

Decycling preserves being EF1 and being EFX.

EF1 exists (Lipton et al '04)

For all $i \in [n]$ set $X_i \leftarrow \emptyset$. All goods are unallocated.
while there is an unallocated good g **do**
    Invariant: X is EF1 and G_X is acyclic
    allocate g to a source of G_X
    Decycle G_X.
Return X.

Correctness: A source is not envied by anybody.

After adding g to the bundle of the source there might be envy towards the source.But removal of g will remove any envy towards the source.

So allocation is EF1.

EFX with Charity Exists (Chaudhury/Kavitha/M/Sgouritsa '20)

For all $i \in [n]$: $X_i \leftarrow \emptyset$.
$P \leftarrow M$. No good is allocated
while Any $U \in \{U_1, U_2, U_3\}$ is applicable **do**
    Invariant: G_X is acyclic and X is EFX.
    $(X, P) \leftarrow U(X, P)$.
    Progress: in new X, no agent is worse off and at least one is better off.
    Decycle G_X.
Return (X, P).

We have three update rules U_1, U_2, and U_3.

As long as one of the rules is applicable, we apply one to update the current allocation and the pool of unallocated goods.

Termination follows from monotonicity.

Update Rule U_1

Precondition: Some agent i envies P.

Action: Give a carefully chosen subset of P to a carefully chosen agent envying P.

Recall: An agent strongly envies a set S of goods if it envies a proper subset of S.

Minimal Envied Subset

```
Z ← P.     Comment: Z is envied by some agent.
while some agent strongly envies Z do
    let g ∈ Z be such that Z \ g is envied by some agent.
    Z ← Z \ g.
    Invariant: Z is envied by some agent.
```

The algorithm determines a minimal (with respect to set inclusion) subset Z of P that is envied by some agent. Z is envied by some agent, but not strongly envied by any agent.

Update Rule U_1

Rule U_1

Precondition: X is EFX and some agent envies P
$Z \leftarrow$ a minimal envied subset of P;
$i \leftarrow$ an agent that envies Z;
assign Z to i and change the pool to $(P \setminus Z) \cup X_i$
Postcondition: X is EFX

- i is better off and hence does not strongly envy any other agent.
- i is not strongly envied by any other agent since Z is a minimal envied subset of P.

Update Rule U_2

Applicable whenever there is $g \in P$ such that $\langle X_1, \ldots, X_i \cup \{g\}, \ldots, X_n \rangle$ is also EFX.

Remove g from the pool and give it to i.

Can have at most m ($= |M|$) consecutive applications of U_2.

Update Rule U_3

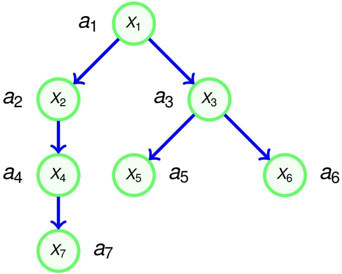
Applicable whenever U_2 is not applicable and $|P| \geq n$.

Warmup: G_X has only one source, say s

- let g be an unallocated item.
- let Z be a minimal envied subset of $X_s \cup g$ and let i be an agent that envies Z.
- let $s = a_0 \to a_1 \to \ldots \to a_k = i$ be a path from s to i in G_X.
- assign X_ℓ to $a_{\ell-1}$ for $1 \leq \ell \leq k$ and assign Z to i.
- remove g from the pool and add $(X_s \cup g) \setminus Z$ to the pool.

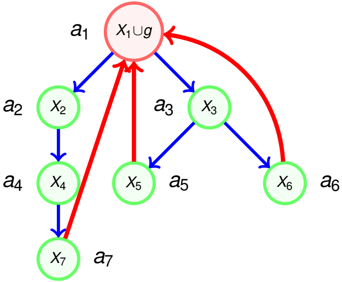
Correctness: The new allocation is EFX since all agents on the cycle are better off and since nobody strongly envies Z.

And Visually

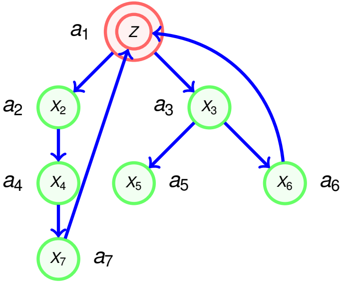


And Visually

Pick any $g \in P$

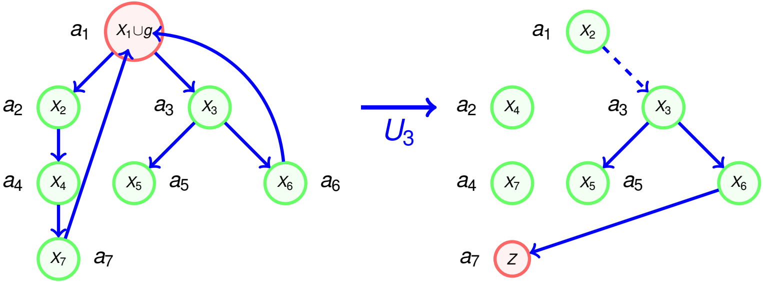


And Visually

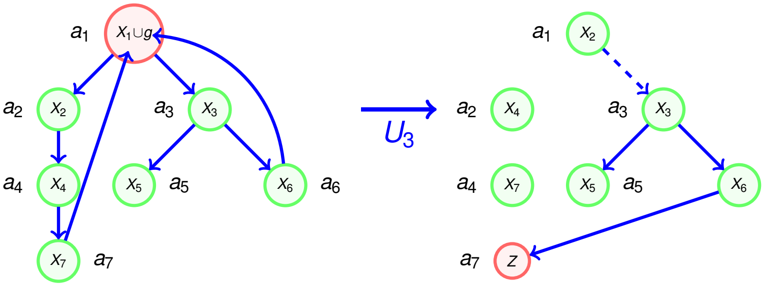


reduce $X_1 \cup g$ to Z until strong envy is gone. a_7 and a_6 still envy

And Visually

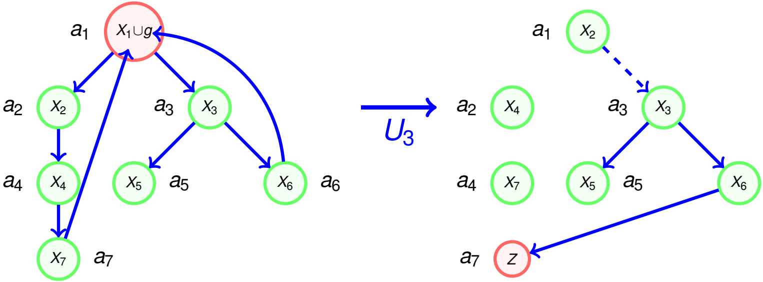


And Visually



Throw $(X_1 \cup g) \setminus Z$ back to P

And Visually

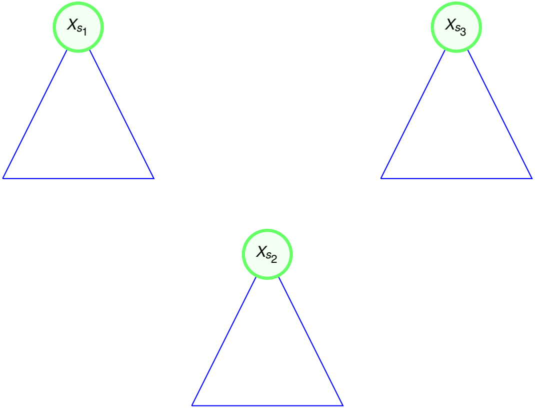


New allocation is EFX and valuation vector improves

Key Idea: Reduce the conflicted bundle ($X_1 \cup g$) and allocate to a most envious agent!

Update Rule U_3

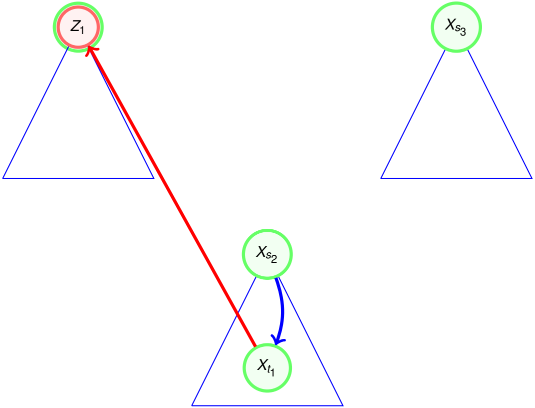
When G_X has multiple sources: Triangles indicate the reachability sets of each source; may overlap.



Update Rule U_3

When G_X has multiple sources: Triangles indicate the reachability sets of each source; may overlap.

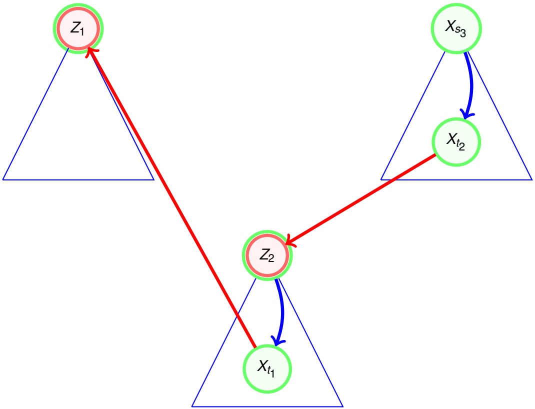
Pick any $g_1 \in P$, let Z_1 be a minimal-envied subset of $X_{s_1} \cup g_1$ and let t_1 be an agent that envies Z_1.



Update Rule U_3

When G_X has multiple sources: Triangles indicate the reachability sets of each source; may overlap.

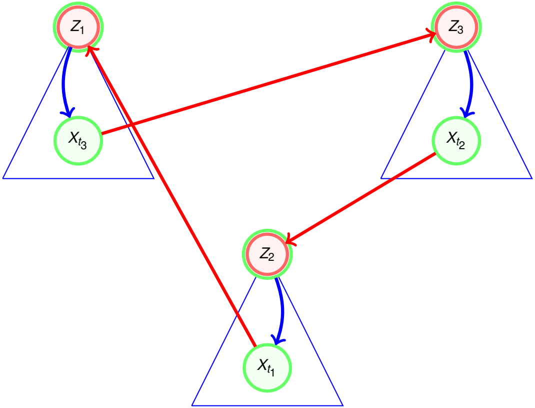
Pick any $g_2 \in P$, let Z_2 be a minimal-envied subset of $X_{s_2} \cup g_2$ and let t_2 be an agent that envies Z_2.



Update Rule U_3

When G_X has multiple sources: Triangles indicate the reachability sets of each source; may overlap.

Pick any $g_3 \in P$, let Z_3 be a minimal envied subset of $X_{s_3} \cup g_3$ and let t_3 be an agent that envies Z_3.



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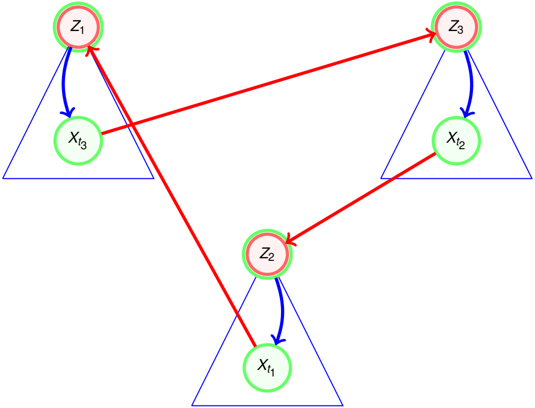

Now we do a cyclic shift of the bundles along the envy cycle. Along the cycle everybody is better off. No strong envy introduced.

Summary

- U_1 is applicable if some agent envies the pool.
- U_2 is applicable if an unallocated good can be allocated to source without destroying EF1.
- U_3 is applicable if U_2 is not applicable and $|P| \geq n$.

EFX with Charity

There exists a partition $\langle X_1, X_2, \ldots, X_n, P\rangle$ of M such that,

- $\langle X_1, X_2, \ldots, X_n\rangle$ is EFX,
- For every i, we have $v_i(X_i) \geq v_i(P)$ and
- $|P| < n$.

Sublinear Charity (Chaudhury/Garg/M/Mehta/Misra, EC ’21)

EFX with Sublinear Charity

For any $\varepsilon \in (0, 1/2)$ there exists a partition $\langle X_1, X_2, \ldots, X_n, P\rangle$ of M such that,

- $\langle X_1, X_2, \ldots, X_n\rangle$ is $(1-\varepsilon)$-EFX,
- For every i, we have $v_i(X_i) \geq v_i(P)$ and
- $|P| \leq 8(n/\varepsilon)^{4/5}$. (improved to $O((n/\varepsilon)^{2/3}$, arXiv)

Definition of $(1-\varepsilon)$-EFX:

for all i and j: $(1-\epsilon)v_i(X_j \setminus g) \leq v_i(x_i)$ for all $g \in X_j$.

EFX for Three Agents, Additive Valuations

When the algorithm for EFX with charity stops and returns an allocation X, there are at most 2 unallocated goods.

How can we allocate the remaining goods? There is a considerable complication.

Fact

There exists a partial EFX allocation X and an unallocated good g, such that in every complete EFX allocation some agent is worse off.

We must allow update rules, that worsen the fate of some agent.

But our termination argument relied on the monotonicity of the update rules.

A Potential Function

Potential Function $\Phi(X)$

- Label the agents arbitrarily as a,b and c.
- $\Phi(X) = \langle v_a(X_a), v_b(X_b), v_c(X_c) \rangle$.

Lemma

Given any partial EFX allocation X and an unallocated good g, there exists another partial EFX allocation X' such that $\Phi(X') >_{lex} \Phi(X)$, i.e.,

- improve a or
- do not worsen a and improve b or
- do not worsen a and b and improve c

and this implies the theorem.

Proof is a 15 page tedious case analyis; Bhaskar pushed it through. We have since simplified the proof to three pages.

Summary

For all $i \in [n]$ set $X_i \leftarrow \emptyset$.
while there is an unallocated good g, **do**
    Invariant: X is EFX.
    $X \leftarrow U(X, g)$ by some update rule U.
    Fact: $\Phi(\text{new } X) > \Phi(\text{old } X)$
Return X.

$n \geq 4$ or General Valuations

Additional ideas are needed.

Partial Progress: General valuations and $n = 3$?

Akrami/Chaudhury/M unpublished: $n = 3$ and two valuations may be arbitrary.

Nash Social Welfare (NSW)

Partition the goods into n bundles $X_1, X_2, \ldots, X_n$ so as to maximize

$$NSW := \left(\prod_i v_i(X_i)\right)^{1/n} \qquad \text{geometric mean.}$$

NSW-optimal allocation is

- invariant under scaling valuations,
- EF1 (Caragiannis/Kurokawa/Moulin/Procaccia/Shah/Wang), and
- EFX for two-valued valuations, i.e., $v_i(g) \in \{1, s\}$ for all i and g.
- Making values more equal increases NSW, since $((a+b)/2)^2 \geq a \cdot b$.
- Shifting value from the rich to the poor increases NSW.

The Border between P and NP

- Finding NSW-optimal allocation is NP-complete in general
  - even for two agents and identical evaluations (reduction from subset sum).
  - for three-valued valuations $v_i(g) \in \{0, 1, s\}$ (reduction from 3d-matching).
  - Even hard to approximate.

- What can be done efficiently?
  - 1.45-approximation algorithm (Barman/Rohit/Vaish)
  - zero-one valued is in P; $v_i(g) \in \{0, 1\}$

Question: Can we delineate the border more precisely?

How about two-valued? $v_i(g) \in \{1, s\}$ with $s \in \mathbb{Q}$, $s > 1$.

Results for $v_i(g) \in \{1, s\}$ with $s > 1$

Two-valued integral is in P. For example, $s = 2, 3, 4, 5, \ldots$

Two-valued is NP-complete and APX-hard if

- $s = p/q$, p and q relatively prime and
- $q \geq 3$,
- for example, $s = 4/3, 5/3, 5/4, \ldots$ (Reduction from multi-dimensional matching)

Two results above appeared in AAAI '22.

This leaves the odd multiples of $1/2$.

- Problem is in P (Akrami, Chaudhury, Hoefer, M, Schmalhofer, Shahkarami, Vermande, van Wijland, unpublished, will be on arxive in four weeks).

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Dichotomies between Two and Three

P	**NP-complete**
2-SAT	3-SAT
2-disjoint paths in planar graphs	3-disjoint paths in planar graphs
matchings in graphs	3-dimensional matchings

matchings with degree constraints, prescribe $S_v \subseteq \mathbb{N}$ for each v.
Is there $M \subseteq E$ such that $\mathrm{deg}_M(v) \in S_v$ for all v.

only gaps of length ≤ 2,	gaps of length ≥ 3,
e.g., $S(v) = \{1, 3, 5, 7\}$	e.g., $S(v) = \{1, 4, 6\}$

We use a reduction from 3-dim matching for the NP-completenss proof for $s = 4/3, \ldots$, and we use parity matchings in our alg for s an odd multiple of $1/2$.

Why is $s = 3/2$ more difficult than integral s?

- An item is heavy if is heavy for at least one agent, i.e., $v_i(g) = s$ for at least one i, light otherwise.
- For integral s, one can consider the heavy and the light goods separately.
- For $s = 3/2$, optimal allocation of heavy items depends on number of light items.
  Consider two agents with identical evaluations, 2 heavy goods and either 2 or 3 light goods.

  With two light goods $(5/2, 5/2)$, where $5/2 = 3/2 + 1$. Both have an odd number of heavy items.

  With three light goods $(3, 3)$, where $3 = 3/2 + 3/2$ and $3 = 1 + 1 + 1$. Both have an even number of heavy items.

  Example hints at a connection with parity matchings.

Summary

Fair division of indivisible items is a rich subject with many interesting open problems.

- EFX for more than 3 agents and additive valuations
- EFX for more than 2 agents and general valuations (3 agents, two arbitrary)
- NSW: improved approximation algorithms:
  - best upper bound is 1.45,
  - APX-hardness proof for 1.0...7.
- MaxiMin-Share, see M. Ghodsi, M. Hajiaghayi, M. Seddighin, S. Seddighin, H. Yami.
- Chores, see Chaudhury, Garg, Metha.