Finding fair and efficient allocations through competitive equilibrium

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UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Summer School on Game Theory and Social Choice

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Allocation Problem

- Set $[m]$ of m items, each comes in unit supply
- Set $[n]$ of n agents

- Allocation $X = (X_1, \dots, X_n)$, where X_{ij} is the amount of item j allocated to agent i such that $\sum_i X_{ij} = 1, \forall j$

- Each agent i has linear (additive) valuation: $v_i(X_i) = \sum_j v_{ij} X_{ij}$ where v_{ij} is value (utility) from a unit amount of item j

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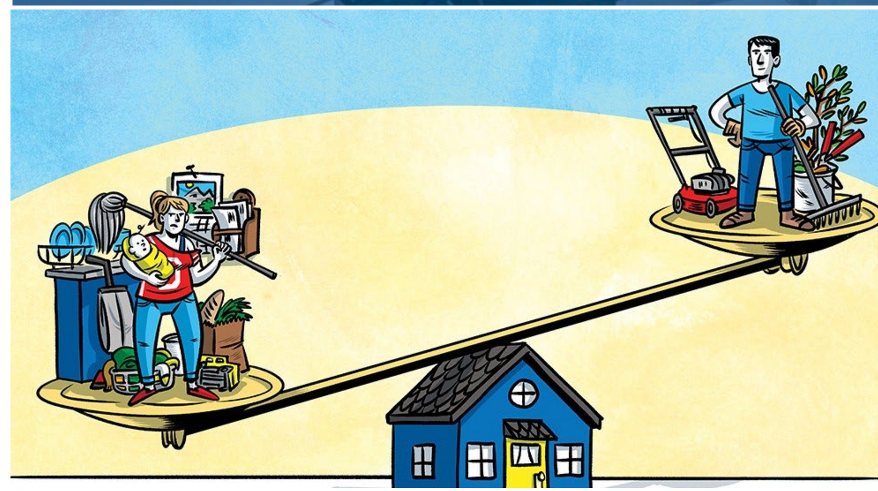
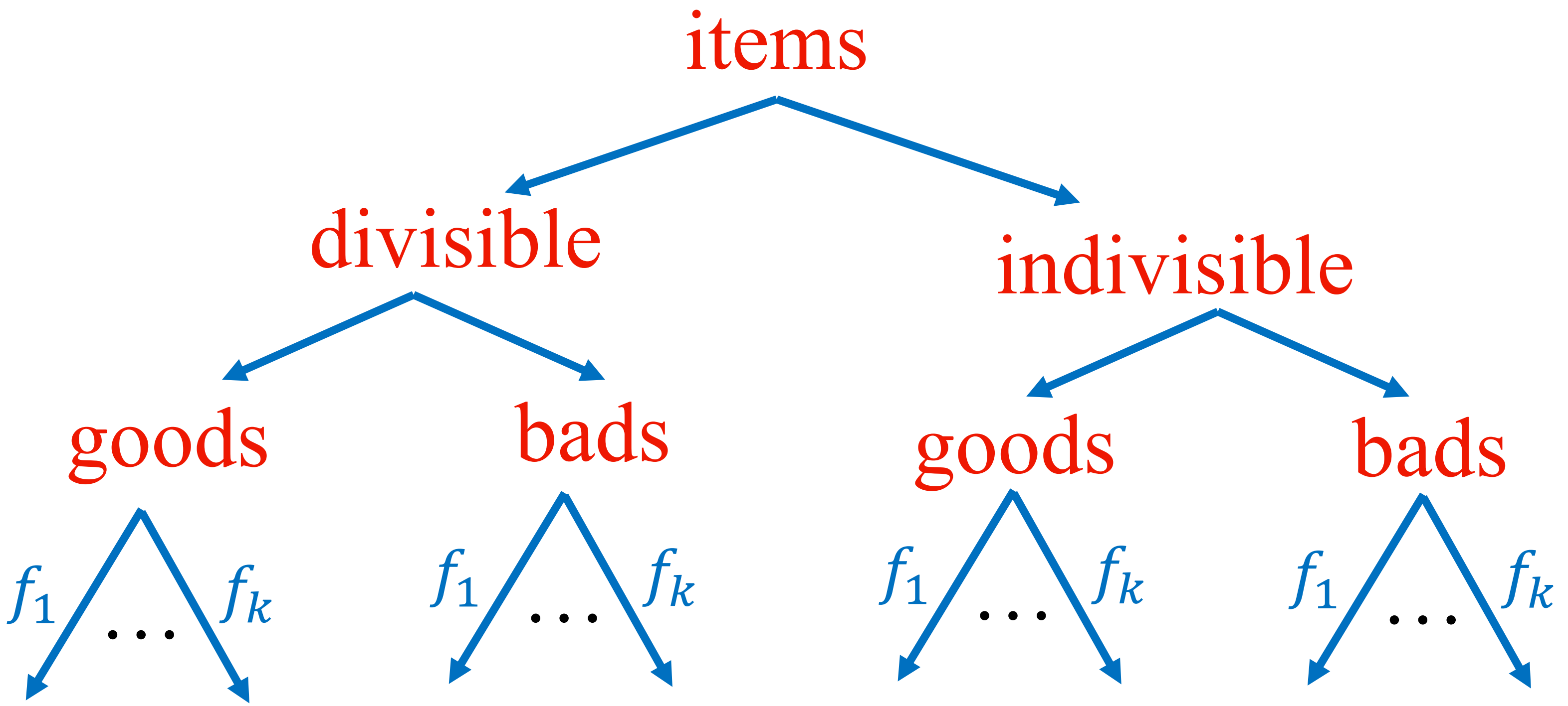
Question: How to "fairly" and "efficiently" allocate items to agents?



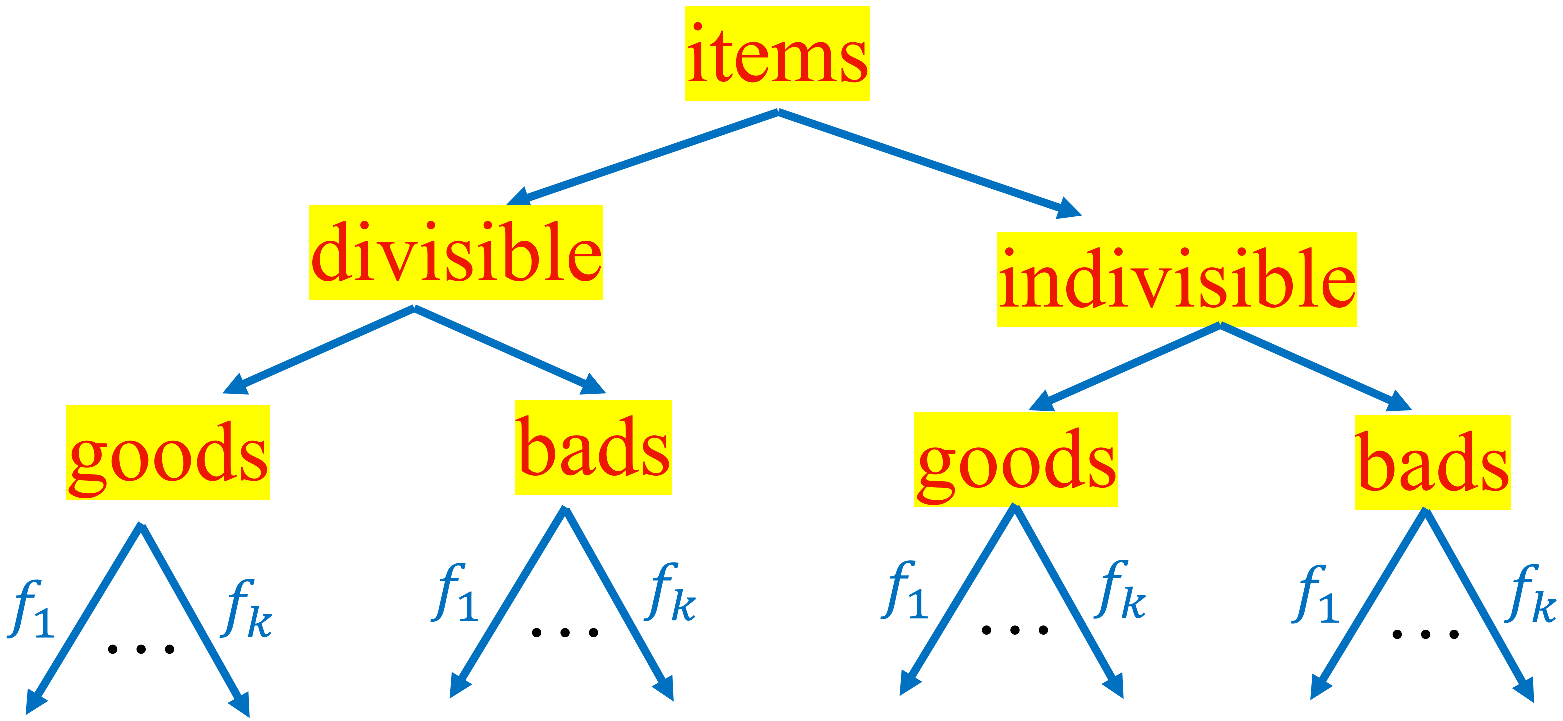

ASSETS
LIABILITIES

Spectrum of Problems



Each f_i = fairness notion

Agenda



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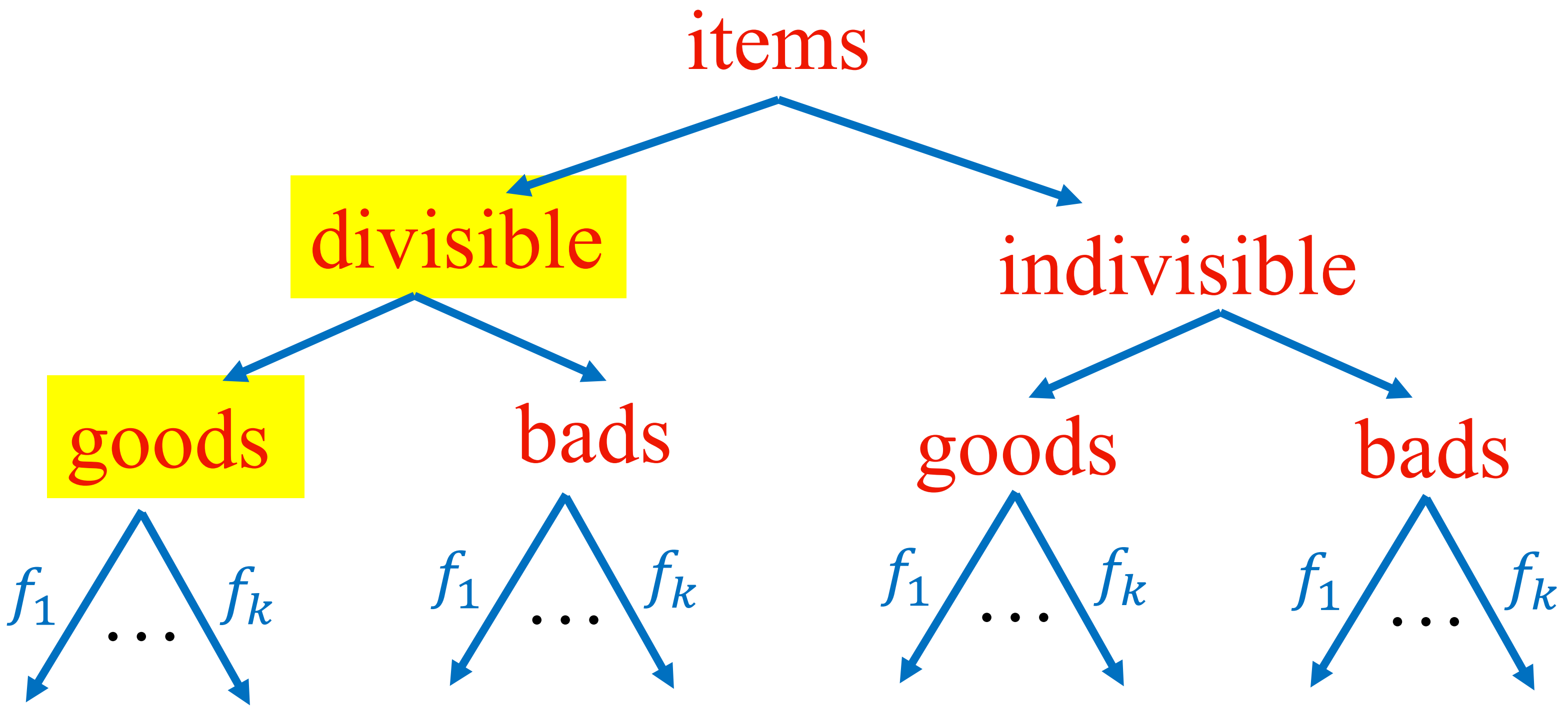


Each f_i = fairness notion

Divisible Goods

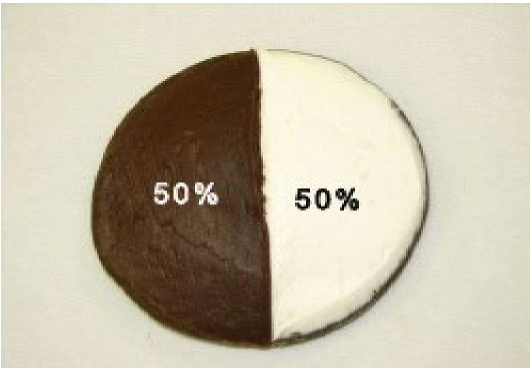
- Set $[m]$ of m divisible goods, each comes in unit supply
- Set $[n]$ of n agents

- Allocation $X = (X_1, \dots, X_n)$, where X_{ij} is the amount of good j allocated to agent i such that $\sum_i X_{ij} = 1, \forall j$

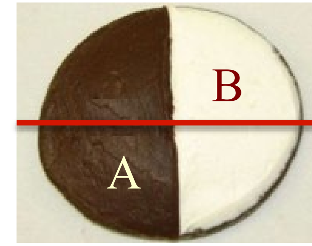
- Each agent i has linear (additive) valuation: $v_i(X_i) = \sum_j v_{ij} X_{ij}$ where v_{ij} is value from a unit amount of good j

Question: How to "fairly" and "efficiently" allocate goods to agents?

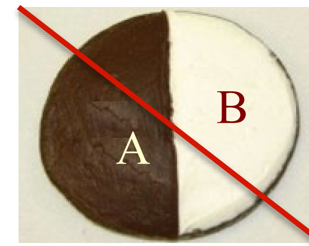
Example: Half moon cookie



(i)



(ii)



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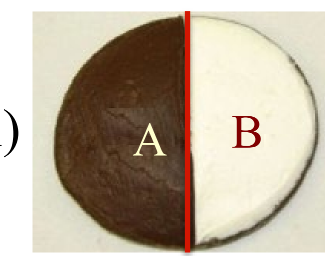


Agreeable (Fair)

Non-wasteful (Efficient)

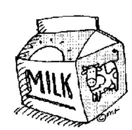
Envy-free: No agent *envies* other's allocation over their own

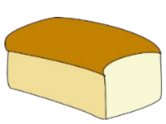
Proportional: Each agent i gets value at least $\frac{v_i([m])}{n}$

[3, 2, 0]
[1/2, 1/2, 1/2]

Allocation in red

[0, 2, 3]
[1/2, 1/2, 1/2]





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Pareto-optimal: No other allocation is better for all.

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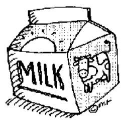

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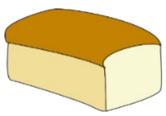

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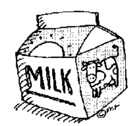

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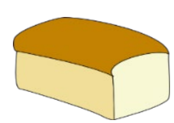
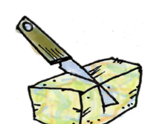


[0, 2, 3]
[0, 1/2, 1]



Agreeable (Fair)

Non-wasteful (Efficient)

Envy-free

Proportional

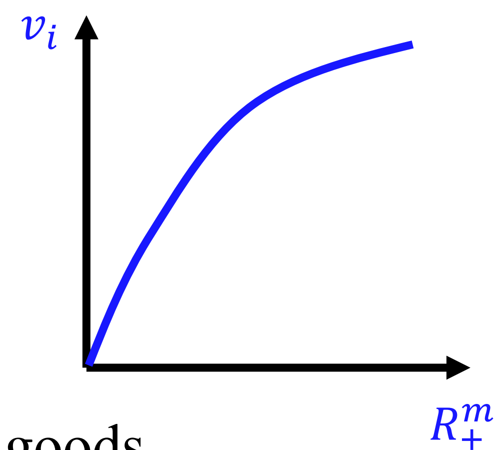
Pareto-optimal

Competitive Equilibrium

(with equal income)

Fisher's Model (1891)

- Set $[m]$ of m divisible goods, each comes in unit supply
- Set $[n]$ of n agents



- Each agent i has
  - budget of B_i dollars
  - valuation function $v_i: R_+^m \to R_+$ over bundle of goods (non-decreasing, non-negative)

Competitive Equilibrium (CE)

Given prices $p = (p_1, \ldots, p_m)$ of goods

- Agent i demands an *optimal bundle*, i.e., affordable bundle that maximizes their utility

$$X_i \in argmax_{x:\, p \cdot x \leq B_i} \, v_i(x)$$

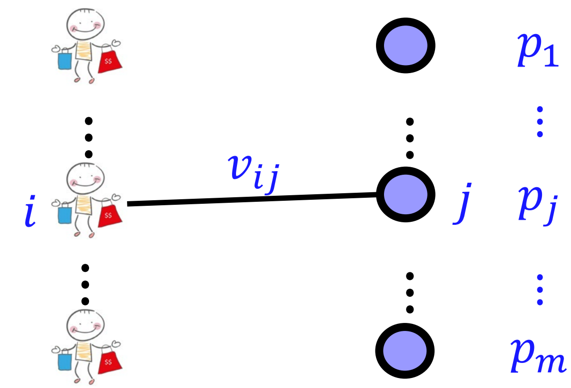
- p is at competitive equilibrium (CE) if *market clears*

Demand = Supply

CE: Linear Valuations

$$v_i(X_i) = \sum_j \boxed{v_{ij}} X_{ij}$$

Utility per unit



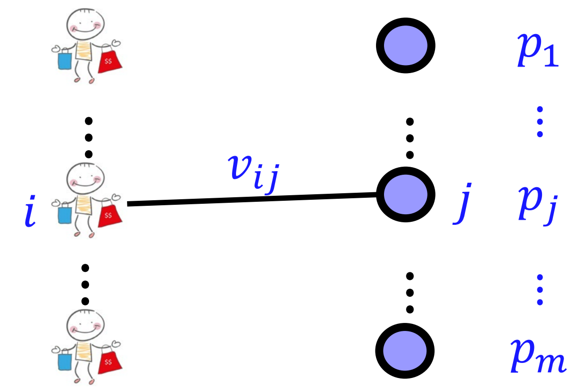
Optimal bundle: can spend at most B_i dollars.

Intuition

spend wisely: on goods that gives max. utility-per-dollar $\frac{v_{ij}}{p_j}$

$$v_i(X_i) = \sum_j \boxed{v_{ij}} X_{ij}$$

Utility per unit



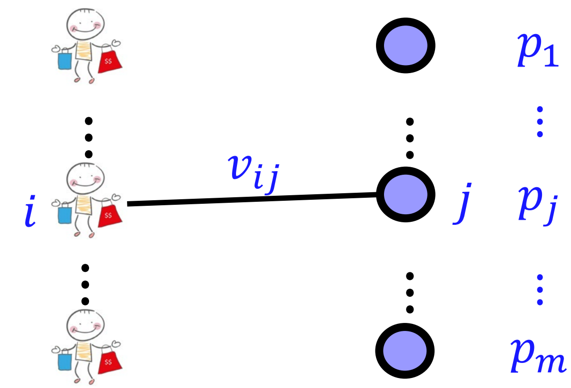
Optimal bundle: can spend at most B_i dollars.

$$\sum_j v_{ij} X_{ij} = \sum_j \boxed{\frac{v_{ij}}{p_j}} \underbrace{(p_j X_{ij})}_{\text{(\$ spent)}} \le \left(\max_k \frac{v_{ik}}{p_k}\right) \sum_j p_j X_{ij} \le \boxed{\left(\max_k \frac{v_{ik}}{p_k}\right)} B_i$$

utility per dollar
(bang-per-buck)

MBB
Maximum
bang-per-buck

$$v_i(X_i) = \sum_j \underbrace{\boxed{v_{ij}}}_{\text{Utility per unit}} X_{ij}$$



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utility per dollar
(bang-per-buck)

MBB
Maximum
bang-per-buck

$=$ **iff**

1. Spends all of B_i.
   $(p.X_i) = B_i$

2. Only on MBB goods
   $X_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = MBB$

CE Characterization

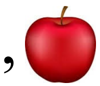
Prices $p = (p_1, \dots, p_m)$ and allocation $X = (X_1, \dots, X_n)$ are at equilibrium iff

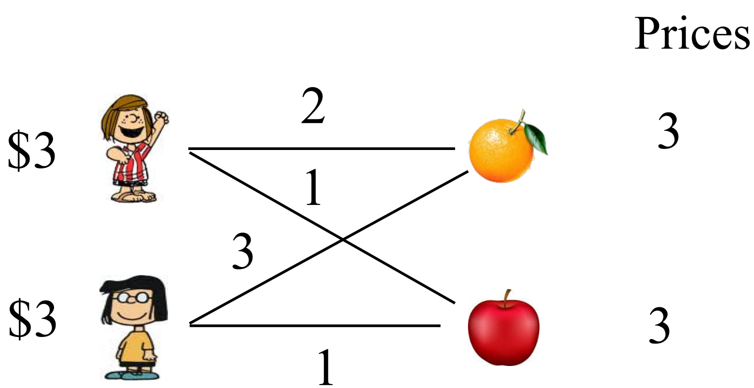
- Optimal bundle: For each agent i
  - $p \cdot X_i = B_i$
  - $X_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_k \frac{v_{ik}}{p_k}$, for all good j

- Market clears: For each good j,

$$\sum_i X_{ij} = 1$$

Example

- 2 Agents (, ), 2 Goods (, ) with unit supply
- Each agent has budget of \$3 and a linear utility function



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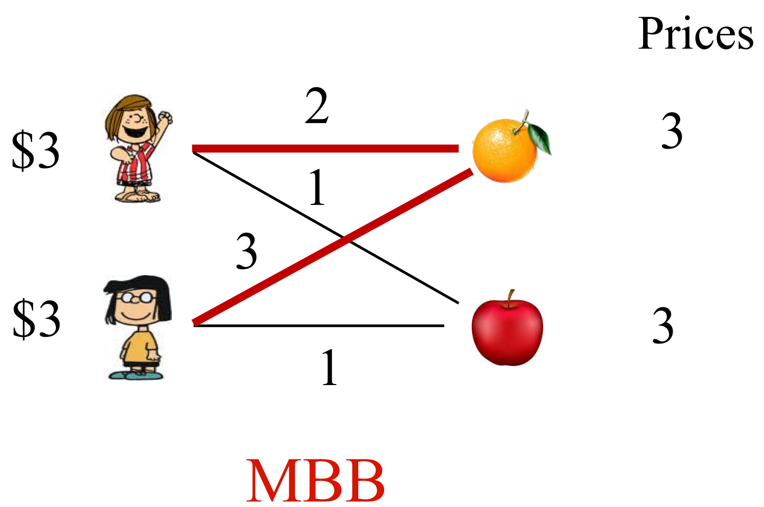
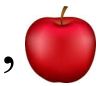


Demand ≠ Supply

Not an Equilibrium!

Example

- 2 Agents (, ), 2 Goods (4 , ) with unit supply
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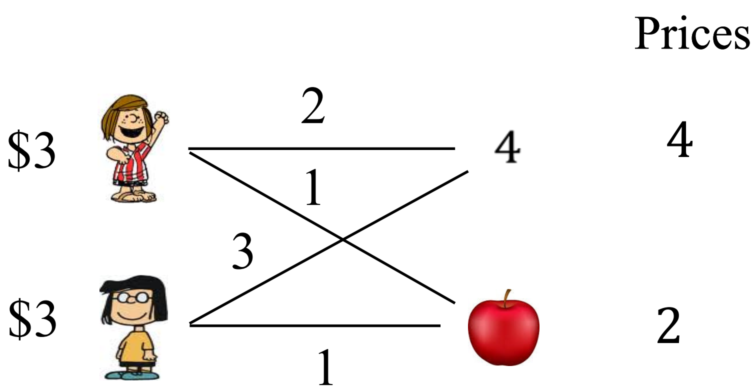

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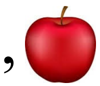
- 2 Agents (, ), 2 Goods (4 , ) with unit supply
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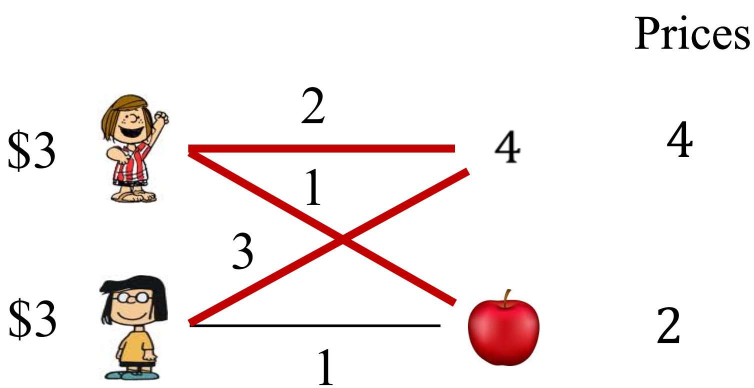


MBB

Demand = Supply

Equilibrium!

Efficiency: Pareto optimality

- An allocation $Y = (Y_1, Y_2, \dots, Y_n)$ Pareto dominates another allocation $X = (X_1, X_2, \dots, X_n)$ if
  - $v_i(Y_i) \geq v_i(X_i)$, for all agents i and
  - $v_k(Y_k) > v_k(X_k)$ for some agent k

- X is said to be Pareto optimal (PO) if there is no Y that Pareto dominates it

First Welfare Theorem

Theorem: Competitive equilibrium outputs a PO allocation

Proof: (by contradiction)

- Let (p, X) be equilibrium prices and allocations
- Suppose Y Pareto dominates X. That is,

  $v_i(Y_i) \geq v_i(X_i), \forall i \in [n]$, and $v_k(Y_k) > v_k(X_k)$ for some k

- Total cost of Y is $\sum_i (p \cdot Y_i) \leq \sum_j p_j$

- k demands X_k at prices p and not Y_k, because?

- Money *agent i* needs to purchase Y_i? ■

Competitive Equilibrium with Equal Income (CEEI)

Problem: Fairly allocate a set of goods among agents without involving money

- Give every agent *(fake)* \$1 and compute competitive equilibrium!

Envy-Free (EF)

Allocation X is **envy-free** if every agent prefers their own bundle than anyone else's. That is, for each agent i,

$$v_i(X_i) \geq v_i(X_k), \forall k \in [n]$$

Theorem: CEEI is envy-free

Proof: Let (p, X) be a CEEI.

- Since the budget of each agent i is \$1, $(p \cdot X_i) = 1$.
- Can agent i afford agent k's bundle (X_k)?
- But she demands X_i instead. Why?

$$v_i(X_i) \geq v_i(X_k)$$

■

Proportionality (Prop)

Allocation X is **proportional** if every agent gets at least the average of their total value of all goods. That is, for each agent i,

$$v_i(X_i) \geq \frac{v_i([m])}{n}$$

Theorem: CEEI is envy-free

Proof: (EF ⇒ Prop)

- Let (p, X) be a CEEI.
- X is EF. That is, $v_i(X_i) \geq v_i(X_k), \forall k \in [n]$. Sum-up over all k

$$n \cdot v_i(X_i) \geq \sum_k v_i(X_k) = v_i\left(\sum_k X_k\right) = v_i([m])$$

■

Summary

CE allocation is:

- Pareto optimal (PO)

with equal incomes (CEEI)

- Envy-free
- Proportional

Existence of Competitive Equilibrium

- Equilibrium exists under a very general class of utility functions
- For linear valuations, Eisenberg-Gale convex program exactly capture all equilibria:

$$\max \sum_{i \in [n]} B_i \log \sum_{j \in [m]} v_{ij} X_{ij}$$

$$\sum_{i \in [n]} X_{ij} = 1, \qquad \forall j$$

$$X_{ij} \geq 0, \qquad \forall i, j$$

Efficient (Combinatorial) Algorithms

Polynomial time

- Flow based [DPSV08, DM13, D**G**M16]

Strongly polynomial time

- Scaling + flow [Orlin10, Vegh16, **G**V19]

CE Characterization

Prices $p = (p_1, \ldots, p_m)$ and allocation $X = (X_1, \ldots, X_n)$ are at equilibrium iff

- Optimal bundle: For each agent i
  - $p \cdot X_i = B_i$
  - $X_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_k \frac{v_{ik}}{p_k}$, for all good j

- Market clears: For each good j,

$$\sum_i X_{ij} = 1$$

Competitive Equilibrium → Flow

Pirces $p = (p_1, \dots, p_m)$ and allocation $F = (F_1, \dots, F_n)$

$$F_{ij} = X_{ij} p_j \text{ (money spent)}$$

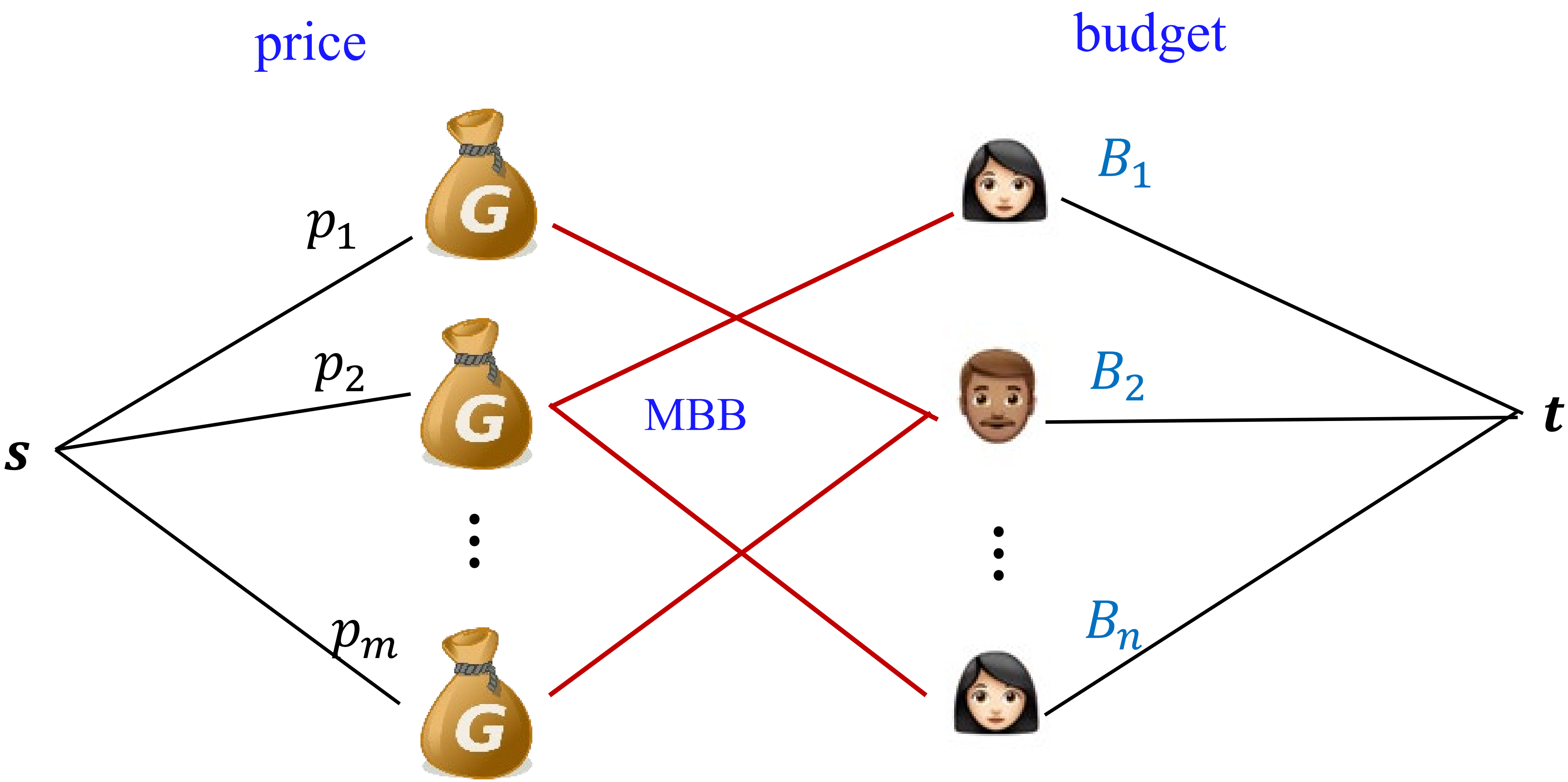
- Optimal bundle: Agent i demands $X_i \in argmax_{x:\, p\cdot x \le B_i} v_i(x)$
  - $\sum_j F_{ij} = B_i$
  - $F_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \boxed{\max_k \frac{v_{ik}}{p_k}}$ for all good j

Maximum bang-per-buck (MBB)

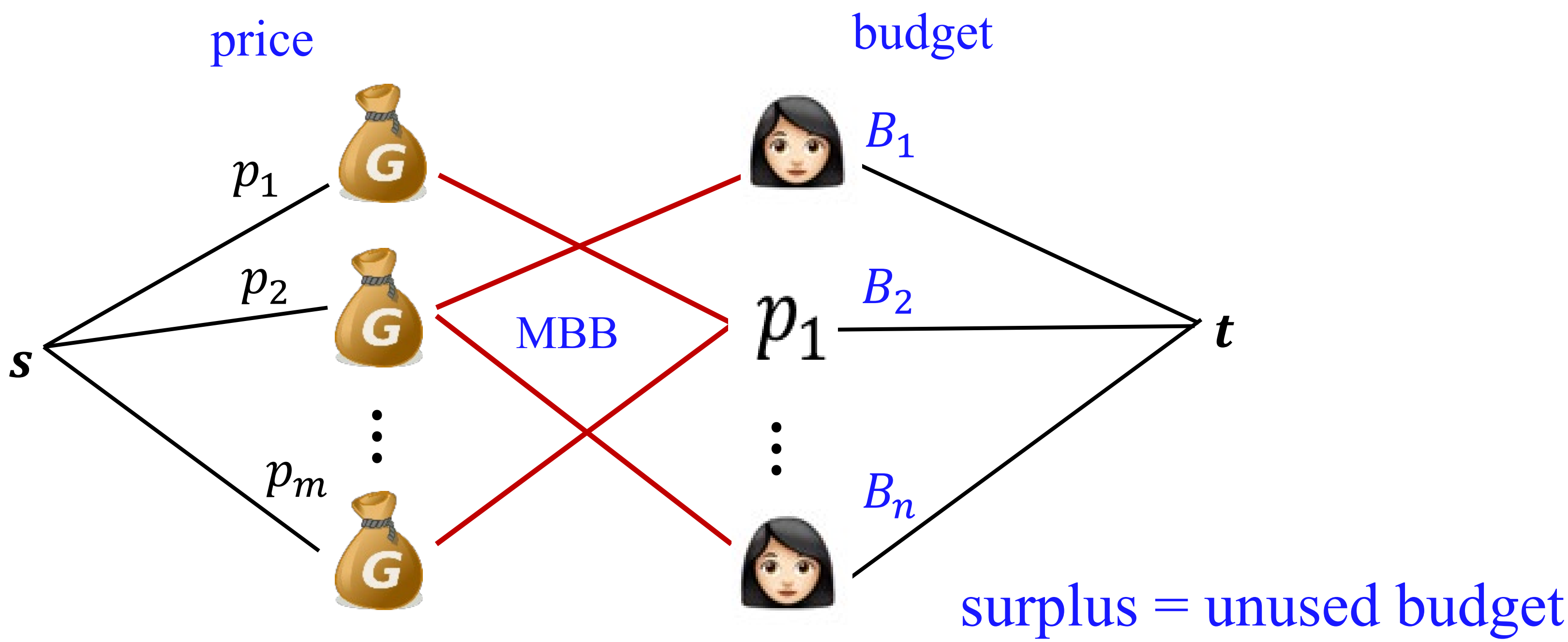
- Market clears: For each good j, demand = supply

$$\sum_i F_{ij} = p_j$$

Network Flow Characterization

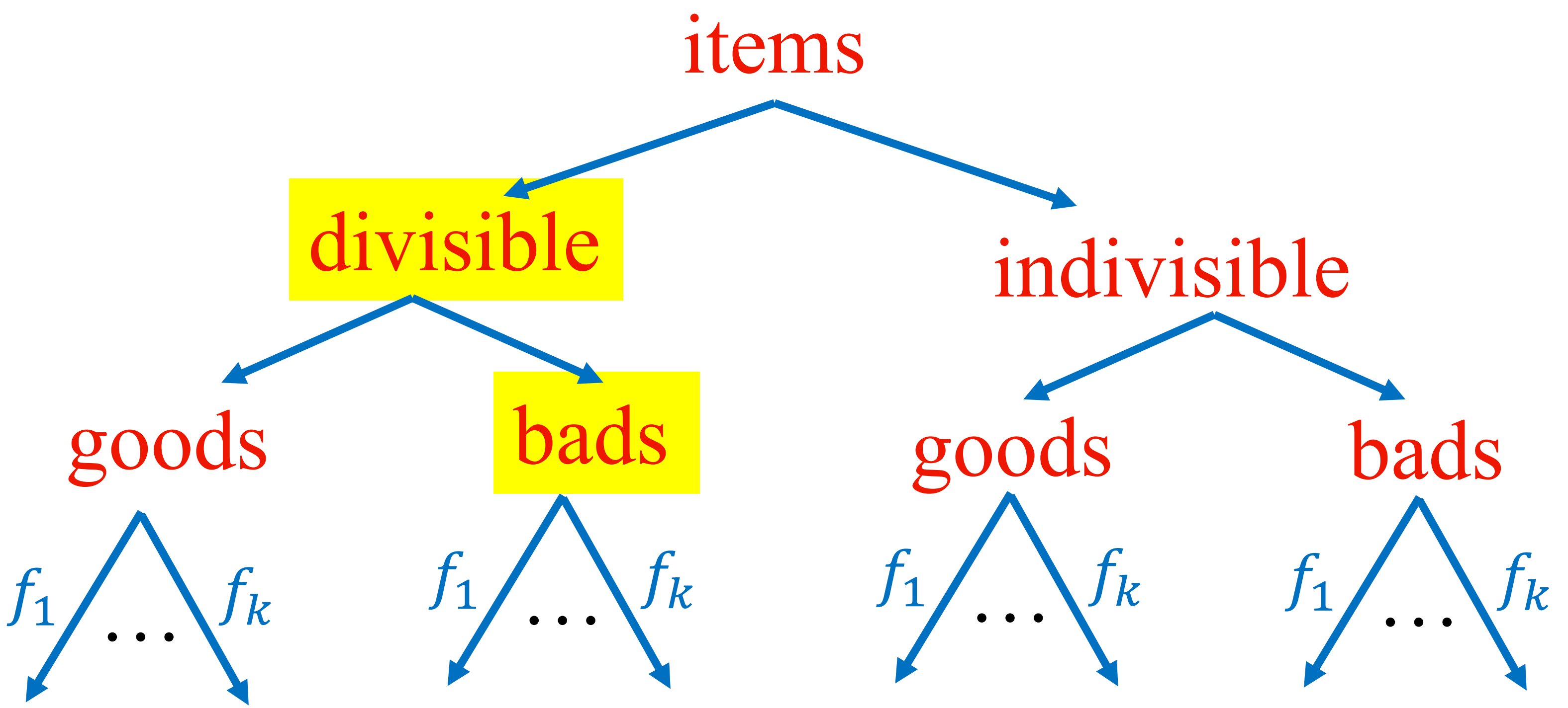


DPSV Algorithm



- Start with tiny p so that max flow = $\sum_j p_j$
- Invariants: prices ⇑, max-flow = $\sum_j p_j$ $\Rightarrow \|\text{surplus}\|_1$ ⇓ $\|\text{surplus}\|_2$ ⇓
- Progress: $\|\text{surplus}\|_2$

Agenda



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Divisible Bads

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- Each agent i has linear (additive) disutility: $d_i(X_i) = \sum_j d_{ij} X_{ij}$ where d_{ij} is disutility from a unit amount of bad j

Question: How to "fairly" and "efficiently" allocate bads to agents?

Agreeable (Fair)

Efficient

Envy-free: No agent *envies* other's allocation over their own

Proportional: Each agent i gets disutility at most $\frac{d_i([m])}{n}$

[100, 1]
[1/2, 1/2]

Allocation in red

[1, 100]
[1/2, 1/2]

Agreeable (Fair)

Envy-free: No agent *envies* other's allocation over their own

Proportional: Each agent i gets disutility at most $\frac{d_i([m])}{n}$

Efficient

Pareto-optimal: No other allocation is better for all.

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[1, 100]
[1, 0]

Agreeable (Fair)

Efficient

Envy-free

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Pareto-optimal

Competitive Equilibrium

(with equal income)

Fisher's Model for bads

- Set $[m]$ of m divisible bads, each comes in unit supply
- Set $[n]$ of n agents

- Each agent i needs to earn
  - at least B_i dollars
  - disutility function $d_i: R_+^m \to R_+$ over bundle of bads

Competitive Equilibrium (CE)

Given prices $p = (p_1, \ldots, p_m)$ of goods

- Agent i demands an *optimal bundle*, i.e., bundle that minimizes their disutility

$$X_i \in argmin_{x:\, p \cdot x \geq B_i}\, d_i(x)$$

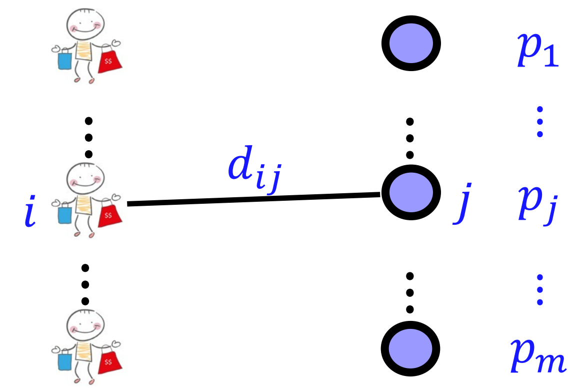
- p is at competitive equilibrium (CE) if *market clears*

Demand = Supply

CE: Linear Valuations

$$d_i(X_i) = \sum_j \boxed{d_{ij}} X_{ij}$$

disutility per unit



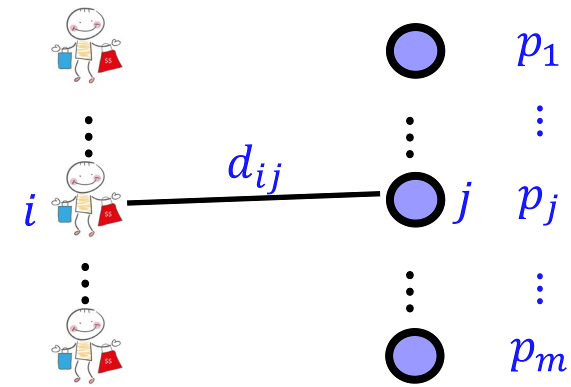
Optimal bundle: needs to earn at least B_i dollar.

Intuition

earn wisely: on bads that gives min. disutility-per-dollar $\frac{d_{ij}}{p_j}$

$$d_i(X_i) = \sum_j \boxed{d_{ij}} X_{ij}$$

Disutility per unit



Optimal bundle: needs to earn at least B_i dollar.

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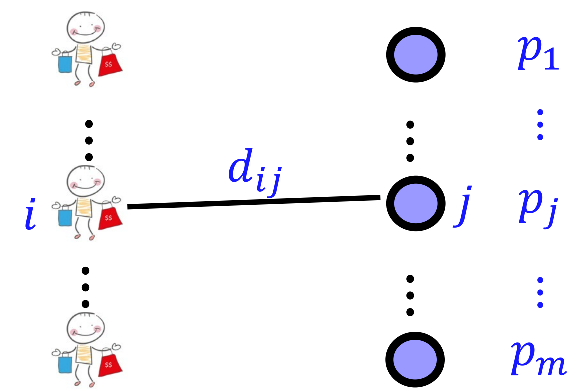
disutility per dollar
(pain-per-buck)

MPB
Minimum
pain-per-buck

$$d_i(X_i) = \sum_j \boxed{d_{ij}} X_{ij}$$

Disutility per unit



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disutility per dollar
(pain-per-buck)

MPB
Minimum
pain-per-buck

=

iff

1. Earns exactly B_i
   $(p.X_i) = B_i$

2. Only on MPB bads
   $X_{ij} > 0 \ \Rightarrow \dfrac{d_{ij}}{p_j} = MPB$

CE Characterization

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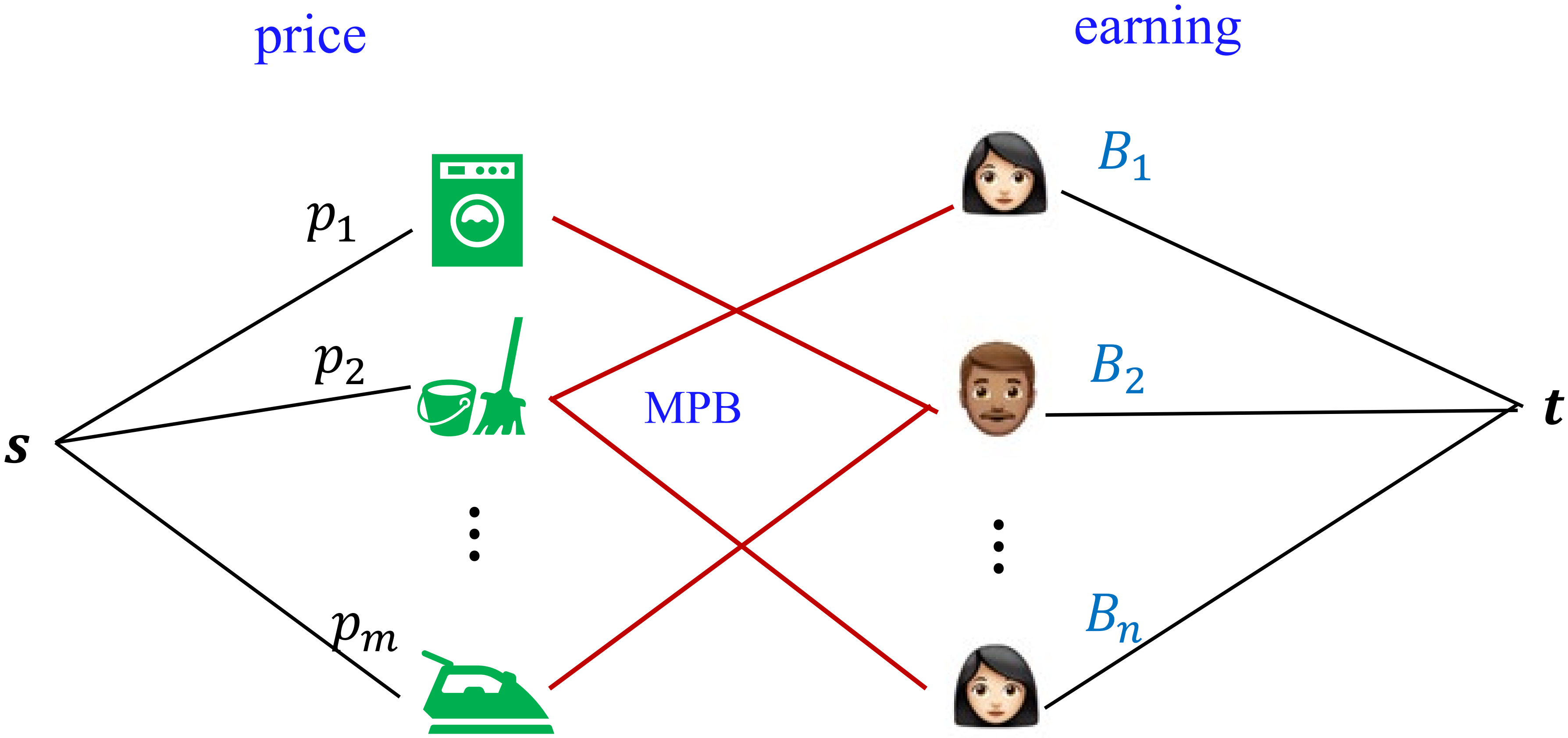
- Market clears: For each bad j,

$$\sum_i X_{ij} = 1$$

Network Flow Characterization



Like the goods case

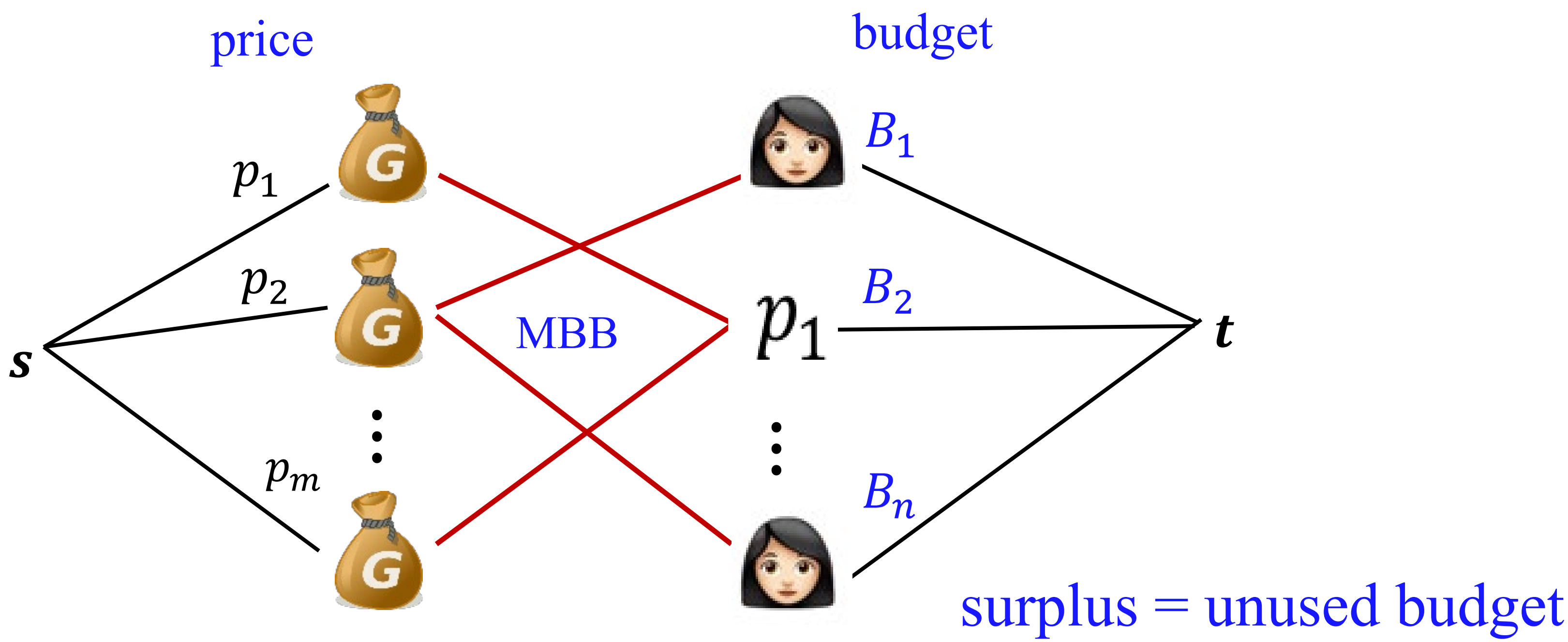
CE allocation in bads is:
- Pareto optimal (PO)

with equal incomes (CEEI)
- Envy-free
- Proportional

Existence: Yes [BMSY17, CGMM21]

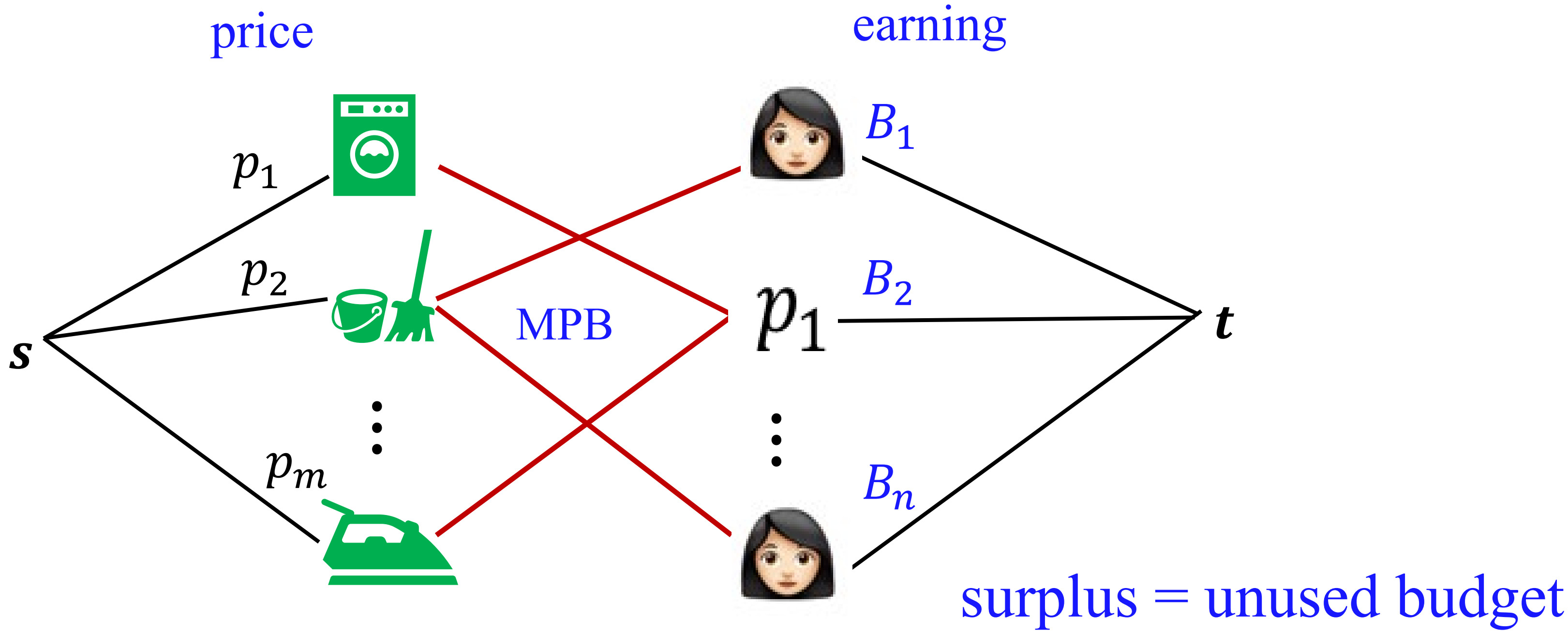
Computation?

DPSV Algorithm [for goods]



- Start with tiny p so that max flow $= \sum_j p_j$
- Invariants: prices $\Uparrow$, max-flow $= \sum_j p_j \implies \|\text{surplus}\|_1 \Downarrow \ \|\text{surplus}\|_2 \Downarrow$
- Progress: $\|\text{surplus}\|_2$

DPSV-type Algorithm for bads?

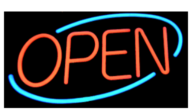


- Start with tiny p so that max flow $= \sum_j p_j$
- Invariants: prices ⇑, max-flow $= \sum_j p_j \implies \|\text{surplus}\|_1$ ⇑ $\|\text{surplus}\|_2$ ⇑
- Progress: ?

Recent breakthroughs

Difficulty: Non-convex and disconnected set of equilibria

- Enumeration-based algorithm [BS19]
- Simplex-like algorithm [C**G**MM21]
- FPTAS for CEEI [C**G**MM22]
- Polynomial-time for bivalued instances ($d_{ij} \in \{1, a\}$) [**G**MQ22]



Complexity of computing exact equilibrium

Special cases: 2-ary ($d_{ij} \in \{1, a_i\}$)

tri-valued ($d_{ij} \in \{1, a, b\}$)

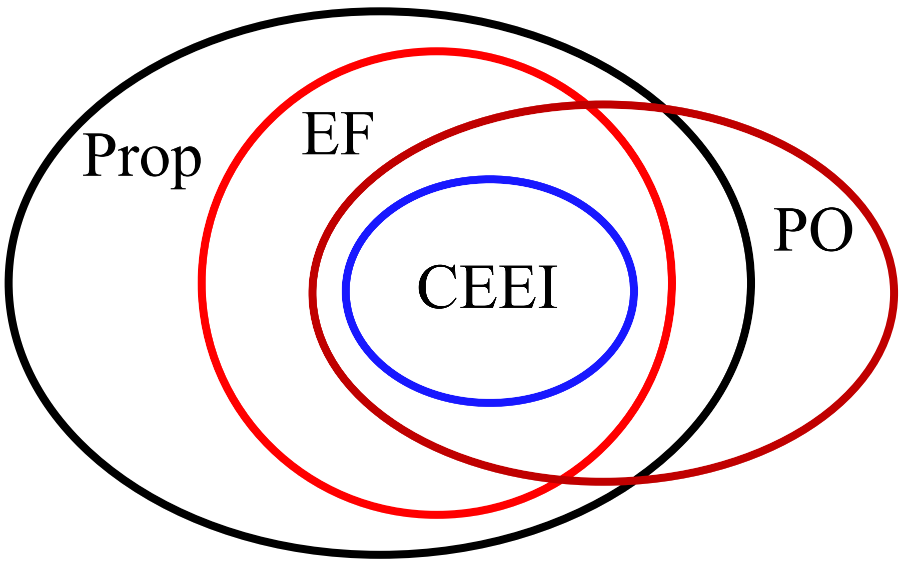
Summary: Divisible Items

- n agents, m divisible items (goods/bads)
- Agent i has linear valuation function $v_i(X_i) = \sum_j v_{ij} X_{ij}$
- Goal: fair and efficient allocation $X = (X_1, \dots, X_n)$

Fairness:

Envy-free (EF)

Proportionality (Prop)



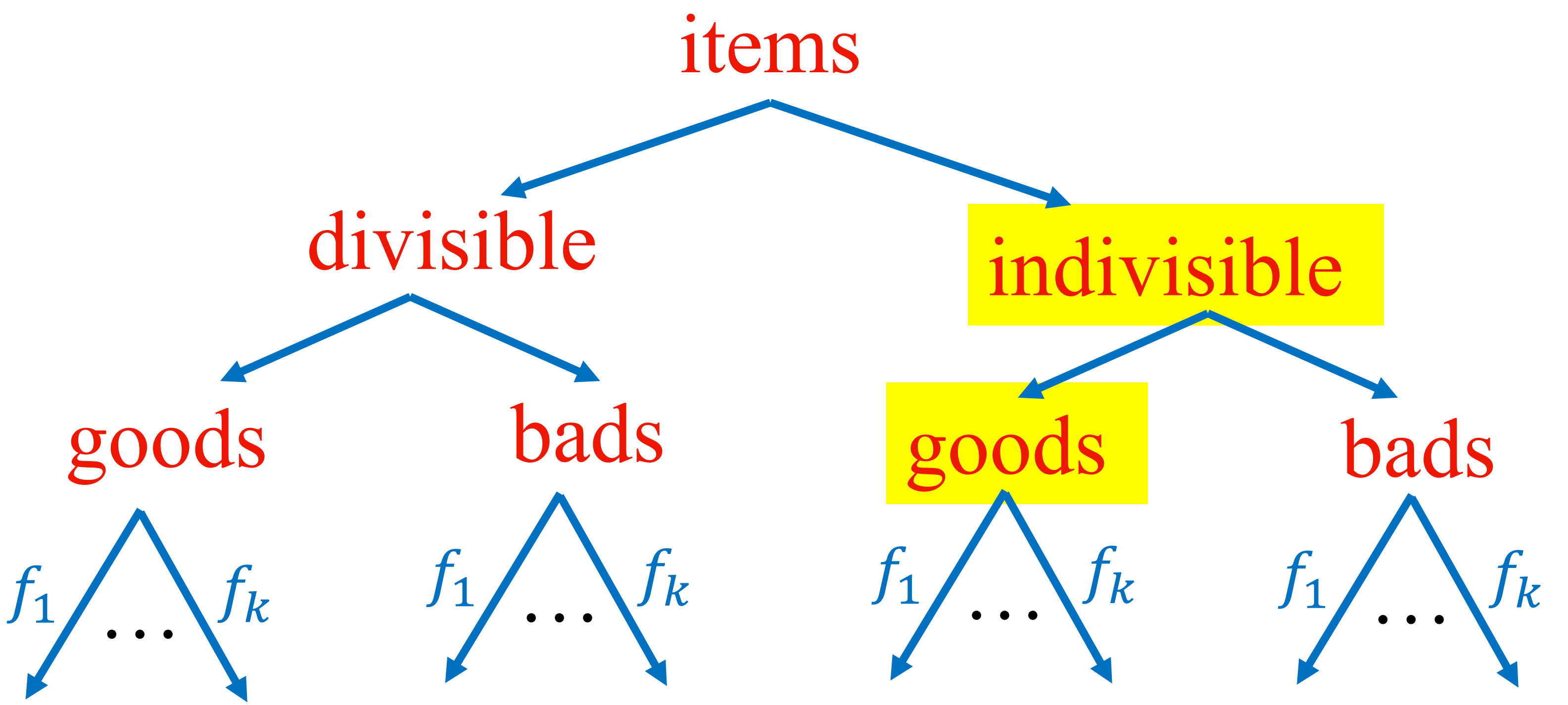
Efficiency:

Pareto optimal (PO)

CEEI Existence: Yes

CEEI Complexity: Open for bads!

Agenda



Each f_i = fairness notion

Indivisible Goods

- n agents, m indivisible goods (like cell phone, painting, etc.)
- Agent i has additive valuation function $v_i(X_i) = \sum_{j \in X_i} v_{ij}$
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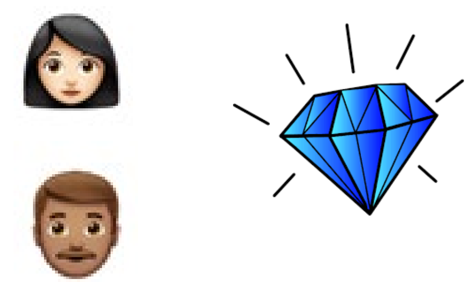
Proportionality (Prop) $\longrightarrow$ Prop1

Envy-free (EF) $\longrightarrow$ EF1

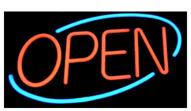


Efficiency:

Pareto optimal (PO)

Existence & Computation

Additive Valuations	**Existence**	**Computation**
Prop1 + PO	✓	Polynomial time [BK19, AMS20, **G**M19]
EF1 + PO	✓ [CKMPSW16]	Pseudo-polynomial time [BKV18, **G**M21]



Complexity of finding an EF1+PO allocation

Indivisible Goods

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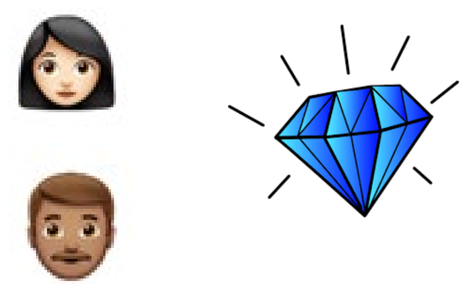
Proportionality up to one good (Prop1)

Proportionality (Prop): Allocation $X = (X_1, \dots, X_n)$ is proportional if each agent gets at least $1/n$ share of all goods:

$$v_i(X_i) \geq \frac{v_i([m])}{n}, \qquad \forall i$$



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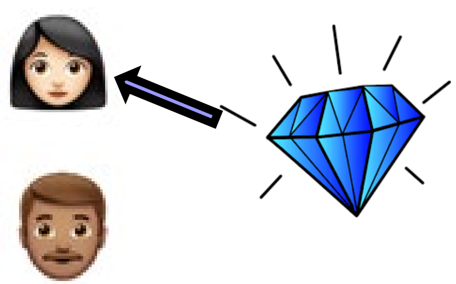
- Prop1: X is proportional up to one good if each agent gets at least $1/n$ share of all goods after adding one more good from outside:

$$v_i(X_i \cup \{g\}) \geq \frac{1}{n} v_i([m]), \qquad \exists g \in [m] \setminus X_i, \forall i$$



Prop1 + PO + … [BK19, **GM19**]

Input: $[n], [m], {v_{ij}}'s$

Step 1 (Prop + PO):

- Assume that all goods are divisible
- (p, X): CEEI
- X is envy-free ⇒ proportional
- we can assume that support of X is acyclic
  - $Support(X) = \{(i,j) \mid X_{ij} > 0\}$

Prop1 + PO + … [BK19, **GM19**]

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  - $Support(X) = \{(i,j) \mid X_{ij} > 0\}$

Step 2 (Rounding):

- In each component of $Support(X)$:
  - Make some agent the root
  - Assign each good to its parent agent

Analysis

Prop1:

- X: CEEI allocation; Y: rounded allocation
- $v_i(X_i) \geq \frac{v_i([m])}{n}$
- Consider X' where $X'_{ij} = \lceil X_{ij} \rceil$
- $v_i(X'_i) \geq v_i(X_i)$
- Relation between Y_i and X'_i?

PO:

- Consider a market $M = [n], [m], v_{ij}'s$, where agent i has budget $\sum_{j \in Y_i} p_j$

Claim: (p, Y) is a CE of M

Indivisible Goods

- n agents, m indivisible goods (like cell phone, painting, etc.)
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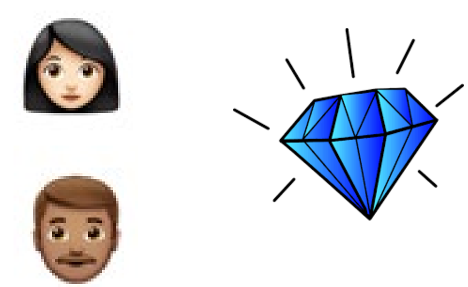

Efficiency:

Pareto optimal (PO)

Envy-Freeness up to One Good (EF1) [B11]

- An allocation $(X_1, \ldots, X_n)$ is EF1 if for every agent i

$$v_i(X_i) \geq v_i\big(X_j \setminus g\big), \;\; \exists g \in X_j, \;\; \forall j$$

That is, agent i may envy agent j, but the envy can be eliminated if we remove a single good from $j's$ bundle

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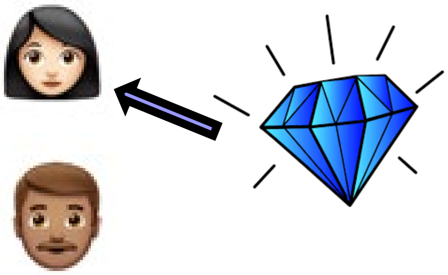
- Existence?



EF1 + PO

Valuations	**Computation**
General Additive	Pseudo-polynomial time [BKV18, **G**M21]
k-ary ($v_{ij} \in \{a_1^i, \dots, a_k^i\}$), k constant	Polynomial time [**G**M21]
Constantly-many agents	Polynomial time [**G**M21]

Complexity of finding an EF1+PO allocation

- Difficulty: Deciding if an allocation is PO is co-NP-hard [KBKZ09]
- Approach: Achieve EF1 while maintaining PO
  - PO certificate: competitive equilibrium!

Competitive Equilibrium (CE)

- m divisible goods, n agents
- Each agent has budget of B_i
- Utility of agent i : $\sum_j v_{ij} X_{ij}$
- p_j: price of item j, F_{ij}: money flow from agent i to good j

Equilibrium (p, F):

1. Optimal bundle: $F_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_k \frac{v_{ik}}{p_k}$

   Maximum bang-per-buck (**MBB**) condition

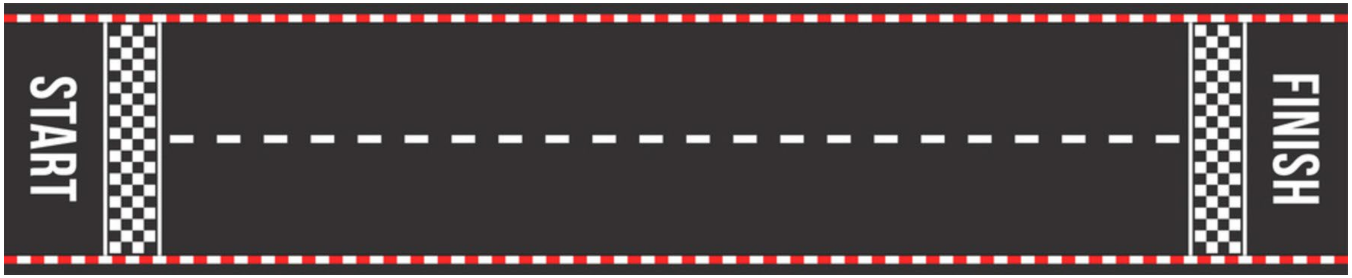
2. Market clearing:

$$\sum_j F_{ij} = B_i , \forall i \qquad and \qquad \sum_i F_{ij} = p_j, \ \forall j$$

EF1+PO [BKV18, **GM21**]

- Approach: Achieve EF1 while maintaining PO

PO



PO
+
EF1

- Starting allocation $X = (X_1, \ldots, X_n)$:
  - Each item j is assigned to an agent with the highest valuation
  - Set price of item j as $p_j = \max_i v_{ij}$
- $p(X_i)$: total price of all goods in $X_i \equiv$ total valuation of i

Example

Claim: (X, p) is (integral) CE when agent i has a budget of $p(X_i)$

Equilibrium (p, F):

1. Optimal bundle **(MBB)**:

$$F_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_k \frac{v_{ik}}{p_k}$$

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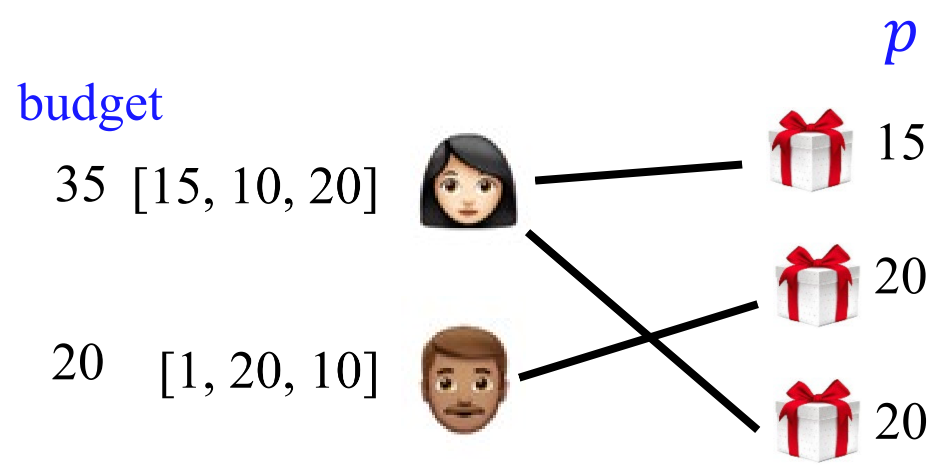

Scaling Valuations with Prices

- Envy-freeness is scale-free
- (X, p): CE
- Let's scale $v_{ij} \leftarrow v_{ij} \cdot \min_k \frac{p_k}{v_{ik}}$

$\Longrightarrow v_{ij} \leq p_j$ and $v_{ij} = p_j$ if $j \in X_i$

Prices can be treated as valuations at CE!

Price-Envy-Free [BKV18]

- (X, p): CE
- X is Envy-Free (EF) if

$$v_i(X_i) \geq v_i(X_j), \qquad \forall i, j$$

$$v_i(X_i) = p(X_i) \qquad p(X_j) \geq v_i(X_j), \qquad \forall i, j$$

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- X is Price-Envy-Free (pEF) if

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- pEF $\Rightarrow$ EF + PO

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- X is Price-Envy-Free (pEF) if

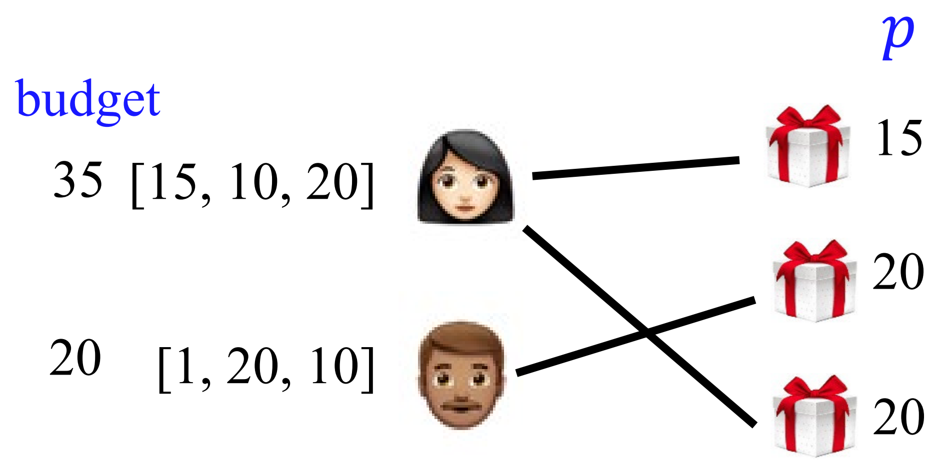
$$p(X_i) \ge p(X_j), \qquad \forall i, j$$

- pEF ⇒ EF + PO

EF?

$$35 = v_1(A_1) \ge v_1(A_2) = 10$$

$$20 = v_2(A_2) \ge v_2(A_1) = 11$$



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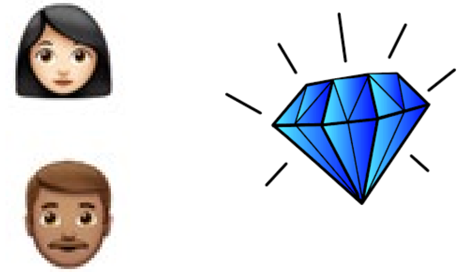


- X is Price-Envy-Free (pEF) if

$$p(X_i) \geq p(X_j), \qquad \forall i, j$$

May not exist!

- pEF $\Rightarrow$ EF + PO

pEF?

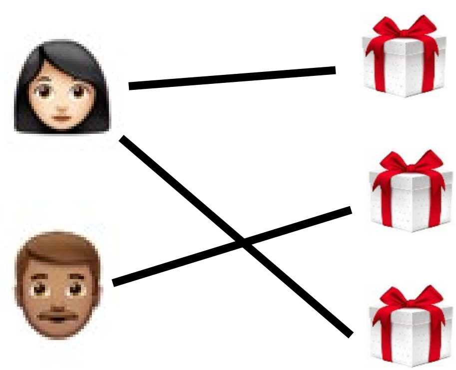
$$35 = p(A_1) \geq p(A_2) = 20$$

$$20 = p(A_2) < p(A_1) = 35$$

budget

35 [15, 10, 20]

20 [1, 20, 10]



p

15

20

20

- (X, p): CE
- X is EF1 if $\quad v_i(X_i) \geq v_i(X_j \setminus g), \quad g \in X_j, \quad \forall i, j$

$$v_i(X_i) = p(X_i) \quad p(X_j \setminus g) \geq v_i(X_j \setminus g), \quad \exists g \in X_j, \quad \forall i, j$$

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- pEF1 ⇒ EF1 + PO

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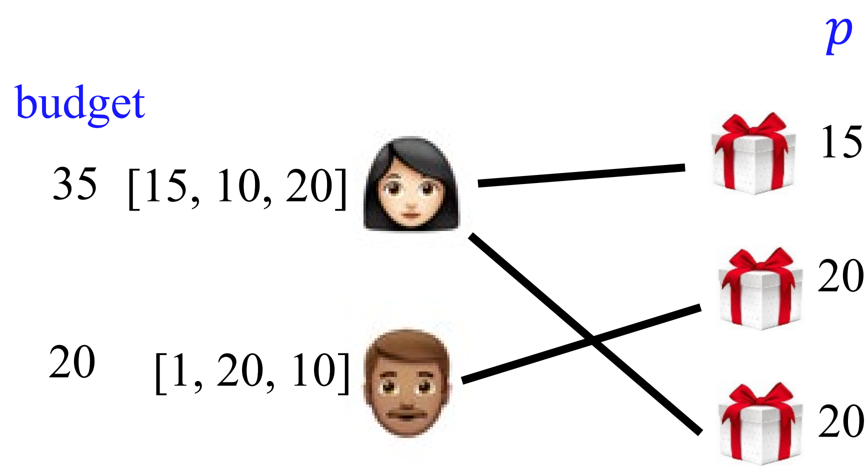
- pEF1 $\Rightarrow$ EF1 + PO

pEF1?

$$35 = p(X_1) > p(X_2 \setminus g_2) = 0$$

$$20 = p(X_2) > p(X_1 \setminus g_3) = 15$$



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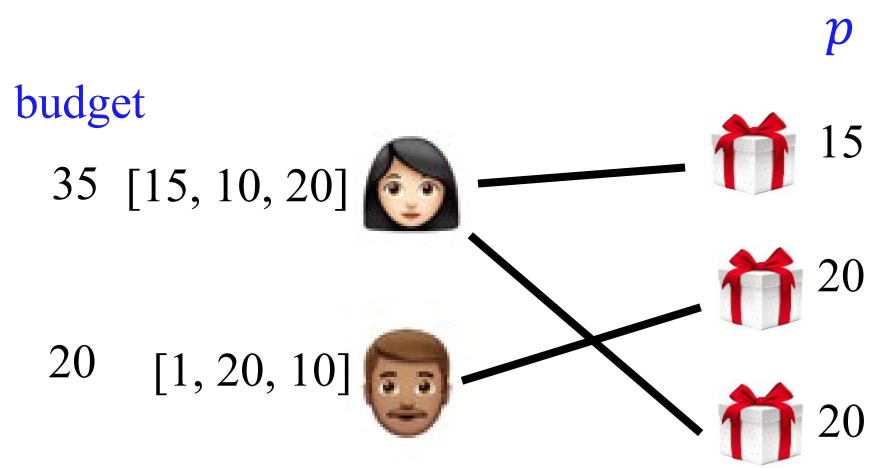
- pEF1 $\Rightarrow$ EF1 + PO



pEF1?

$$35 = p(X_1) > p(X_2 \setminus g_2) = 0$$

$$20 = p(X_2) > p(X_1 \setminus g_3) = 15$$

Theorem [BKV18, **G**M21]: There exists a pseudo-polynomial time procedure to find a pEF1 allocation. Polynomial-time for special cases

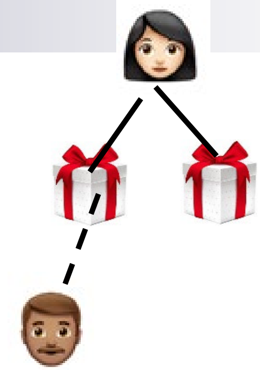
- (X, p): CE
- X is pEF1 if

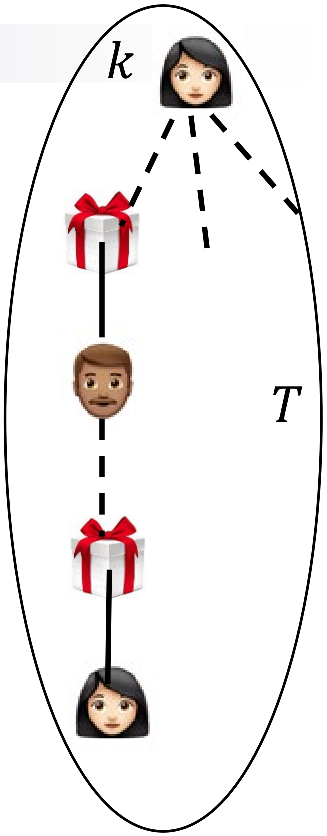
$$p(X_i) \geq p\left(X_j \setminus g\right), \qquad \exists g \in X_j, \qquad \forall i, j$$

- If $\min_i p(X_i) \geq \max_j \min_{g \in X_j} p\left(X_j \setminus g\right)$ then ?

(least spender) (big spender)

Procedure

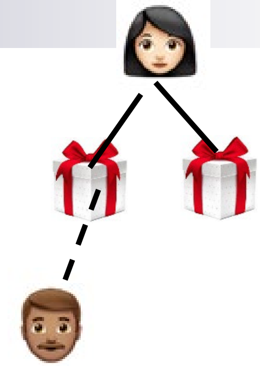




While X is not pEF1

$k \leftarrow \arg\min_i p(X_i)$ //least spender

$T \leftarrow$ Agents and items, k can reach in MBB residual network



While X is not pEF1

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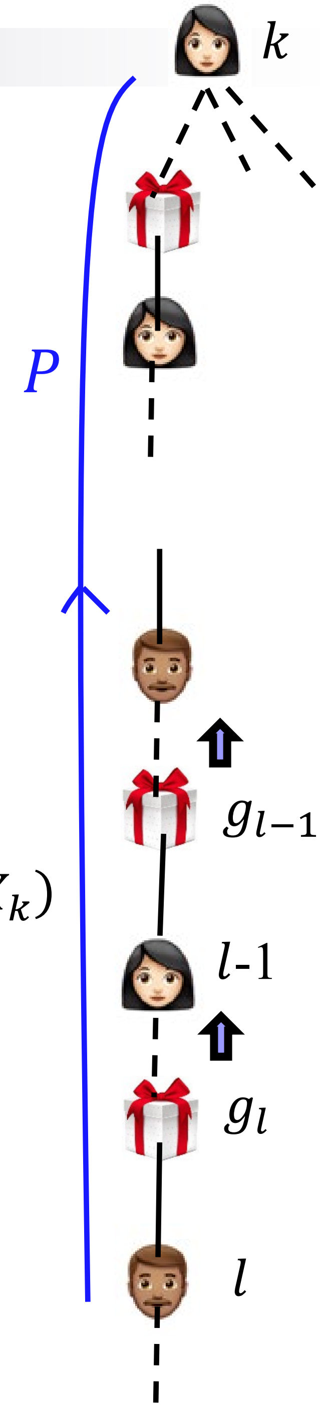
$T \leftarrow$ Agents and items, k can reach in MBB residual network

If k can reach l in T such that $p(X_l \setminus g_l) > p(X_k)$

Pick the nearest such l

$P \leftarrow$ Path from l to k

$X \leftarrow$ Reassign items along P until $p\big((X_j \cup g_{j+1}) \setminus g_j\big) \leq p(X_k)$



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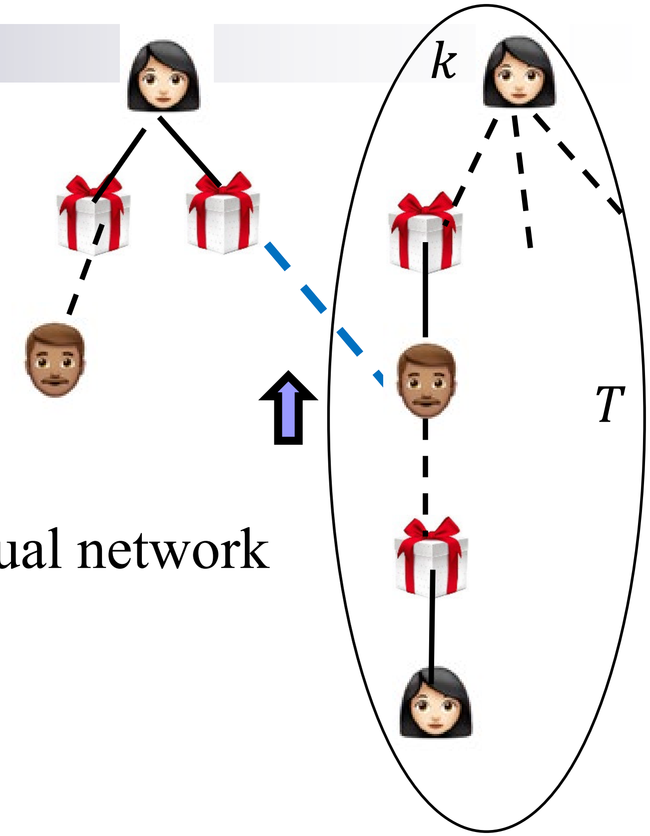
else increase prices of items in T by a same factor until

Event 1: new MBB edge

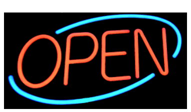
Event 2: k is not least spender anymore

Event 3: A becomes pEF1



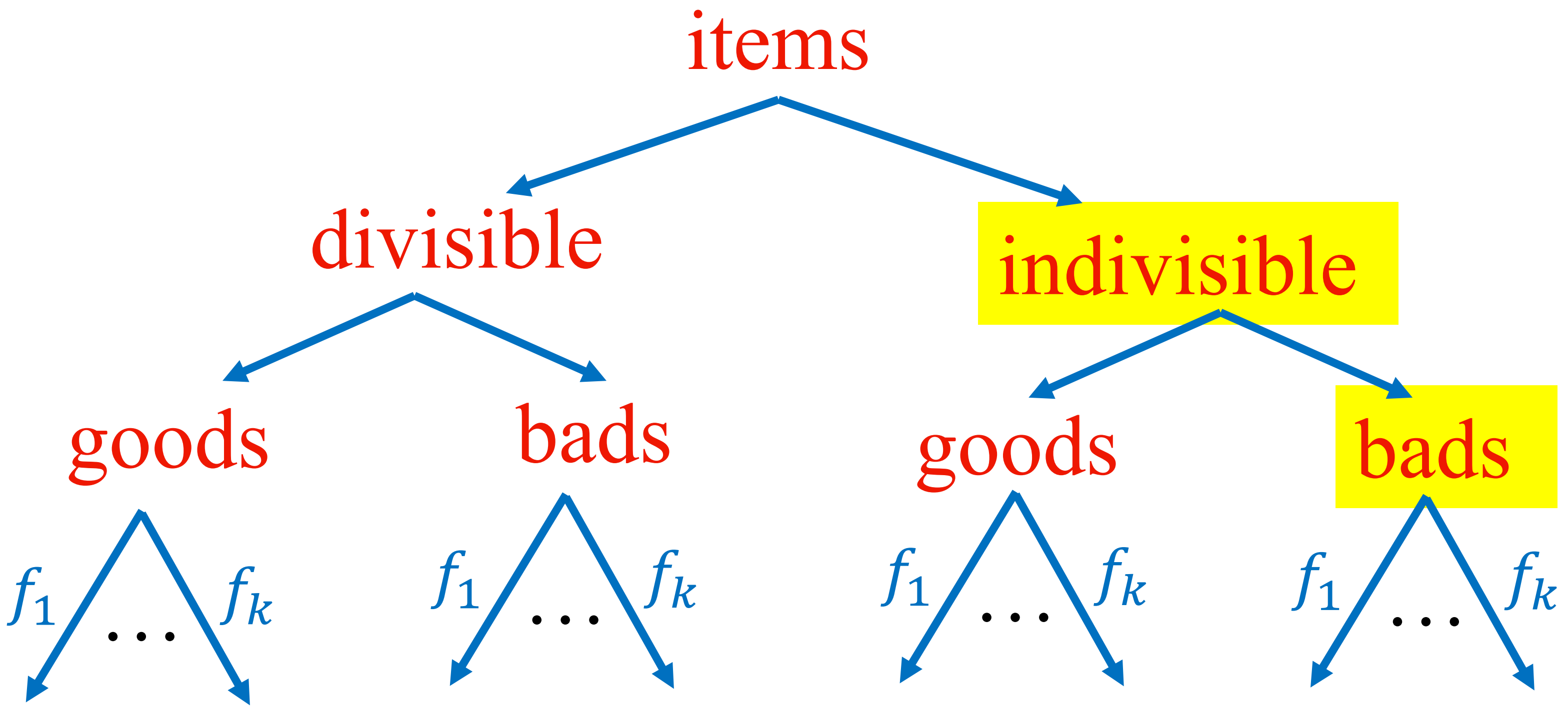
$\min_i p(X_i)$ $\max_j \min_{g \in X_j} p\left(X_j \setminus g\right)$

Claim: The procedure converges to a pEF1 allocation in pseudo-polynomial time!



Complexity of finding an EF1+PO allocation!

Agenda



Each f_i = fairness notion

Indivisible Bads

- n agents, m indivisible bads
- Agent i has additive disutility function $d_i(X_i) = \sum_{j \in X_i} d_{ij}$
- Goal: fair and efficient allocation $X = (X_1, \ldots, X_n)$

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Fairness:

Proportionality (Prop) $\longrightarrow$ Prop1

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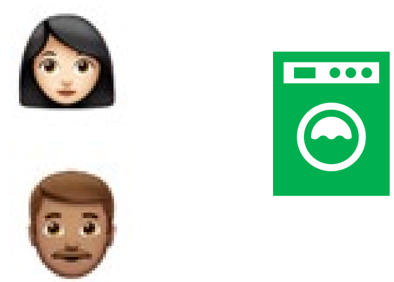

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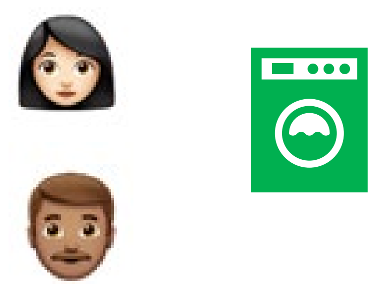
Proportionality (Prop): Allocation $X = (X_1, \dots, X_n)$ is proportional if each agent gets at most $1/n$ share of all bads:

$$d_i(X_i) \geq \frac{d_i([m])}{n}, \qquad \forall i$$



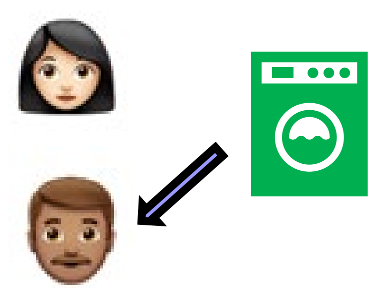
Proportionality up to One Good (Prop1)

Proportionality (Prop): Allocation $X = (X_1, \ldots, X_n)$ is proportional if each agent gets at most $1/n$ share of all bads:

$$d_i(X_i) \geq \frac{d_i([m])}{n}, \qquad \forall i$$

- Prop1: X is proportional up to one bad if each agent gets at most $1/n$ share of all bads after removing one bad from their bundle:

$$d_i(X_i \setminus \{c\}) \leq \frac{1}{n} d_i([m]), \qquad \exists c \in X_i, \forall i$$

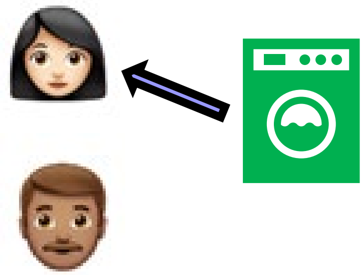


Envy-Freeness up to One Bad (EF1)

- An allocation $(X_1, \ldots, X_n)$ is EF1 if for every agent i

$$d_i(X_i \setminus c) \leq d_i(X_j), \quad \exists c \in X_i, \quad \forall j$$

That is, agent i may envy agent j, but the envy can be eliminated if we remove a single bad from $i's$ bundle



Existence & Computation

Additive Valuations	Existence	Computation
Prop1 + PO	✓	Polynomial time [AMS20]
EF1 + PO	?	
EF1	✓	Polynomial time

EF1 + PO

Valuations	Existence	Computation
General Additive	?	?
Bivalued ($d_{ij} \in \{1, a\}$)	✓	Polynomial time [**G**MQ22, EPS22]
Tri-valued ($d_{ij} \in \{1, a, b\}$) 2-ary ($d_{ij} \in \{1, a_i\}$) 3 agents	?	?

Summary

Fairness:

Proportionality (Prop) $\longrightarrow$ Prop1 PropM PropX MMS

Envy-free (EF) $\longrightarrow$ EF1 EFX

Efficiency:

Pareto optimal (PO) Maximum (Nash) welfare

Major Open Questions

- CEEI for bads: exact complexity
- EF1+PO: exact complexity for goods
- EF1+PO: Existence/Computation for bads

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