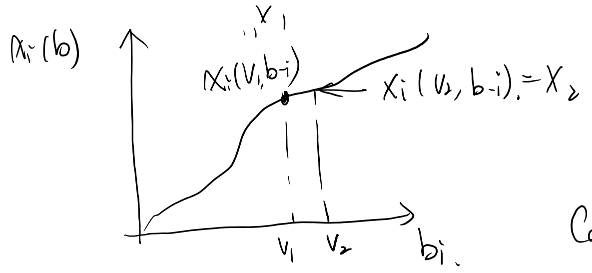
# Summer School - Myerson's Lemma and VCG mechanism

2022年6月21日 11:35

Myerson's lemma. Implementable $\Leftrightarrow$ monotone.

monotone $\Rightarrow$ implementable.

fix $i$, $b_{-i}$. $\qquad x_i(v_2, b_{-i}) \geq x_i(v_1, b_{-i})$




If $(x,p)$ is truthful, we try to find the requirements of $p$.

Case 1: true private value $= v_1$

"bid $v_1$" is better than "bid $v_2$"

$$\underbrace{v_1 \cdot x_i(v_1, b_{-i})}_{x_i(v_1,b_{-i}) = x_1} - \underbrace{p_i(v_1, b_{-i})}_{= p_1} \geq v_1 \cdot \underbrace{x_i(v_2, b_{-i})}_{= x_2} - \underbrace{p_i(v_2, b_{-i})}_{= p_2}$$

$$\rightarrow \boxed{v_1 \cdot x_1 - p_1 \geq v_1 \cdot x_2 - p_2}$$

Case 2. true private value $= v_2$

"bid $v_2$" is better than "bid $v_1$"

$$\rightarrow \boxed{v_2 \cdot x_2 - p_2 \geq v_2 \cdot x_1 - p_1}$$

$$\boxed{v_1 (x_2 - x_1) \leq p_2 - p_1 \leq v_2 (x_2 - x_1).}$$

choose. $v_2 = v_1 + \Delta v$. $\quad (\Delta v > 0)$

$$\frac{v_1 \left( x(v_1 + \Delta v) - x(v_1) \right)}{\Delta v_1} \leq \frac{p(v_1 + \Delta v) - p(v_1)}{\Delta v} \leq \frac{v_2 \left( x(v_1 + \Delta v) - x(v_1) \right)}{\Delta v.}$$

$\Delta v \to 0$.

$$\frac{dp(v)}{dv} = v \cdot \frac{dx(v)}{dv}$$

$$p(v) = \int_0^v z \cdot x'(z) \cdot dz + C$$

$$p_i(v, b_{-i}) = \int_0^v z \, x_i'(z, b_{-i}) \, dz + C.$$

Assumption 1. $x_i(z, b_{-i})$ is derivable w.r.t. $z$.

Assumption 2. $b_i = 0 \Rightarrow p_i(b_i) = 0$.

$\Downarrow$

$C = 0$.

prove $(x, p)$ is truthful.

Goal: $\forall i, \forall b_{-i}, \forall b_i$. "bid $v_i$" is better than "bid $b_i$"

true value ↑ $\Updownarrow$ ↑ arbitrary bid

$$v_i \cdot x_i(v_i, b_{-i}) - p_i(v_i, b_{-i}) \geq v_i \cdot x_i(b_i, b_{-i}) - p_i(b_i, b_{-i})$$

$$p_i(v_i, b_{-i}) - p(b_i, b_{-i}) = \int_{b_i}^{v_i} z\, x'(z)\, dz$$

$$\leq v_i \int_{b_i}^{v_i} x'(z)\, dz \leq v_i (x(v_i) - x(b_i))$$

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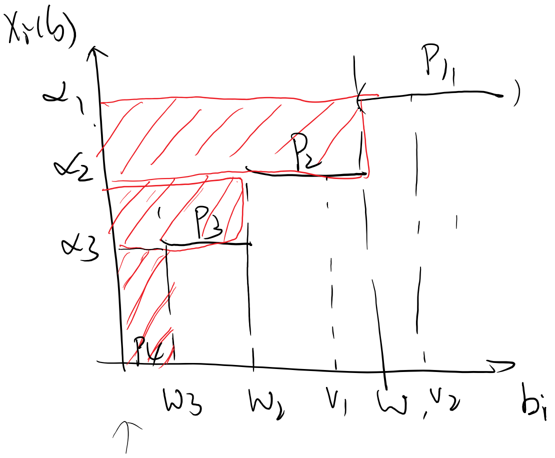
Myerson's Lemma for sponsored search auction.

case $k = 3$.




$$p(v) = \int_0^v z\, x'(z)\, dz.$$

Assume, $P_4 = 0$.

$$v_1(\alpha_1 - \alpha_2) \leq P_1 - P_2 \leq v_2(\alpha_1 - \alpha_2)$$

$$\forall\ v_2 \in [w_1, \infty),\ v_1 \in [w_2, w_1]$$

$$\Rightarrow\ P_1 - P_2 = \underline{w_1 \cdot (\alpha_1 - \alpha_2)}$$

$$P_2 - P_3 = w_2 \cdot (\alpha_2 - \alpha_3)$$

$$P_3 - P_4 = w_3 \cdot \alpha_3$$

$$\begin{cases} P_3 = w_3 \cdot \alpha_3 \\ P_2 = w_3 \cdot \alpha_3 + w_2(\alpha_2 - \alpha_3) \\ \underline{P_1} = w_3 \cdot \alpha_3 + w_2(\alpha_2 - \alpha_3) + w_1(\alpha_1 - \alpha_2) \end{cases}$$

VCG mechanism.

Agent $i$: utility: $u_i(b) = v_i(x(b)) - \underline{p_i(b)}$

$$= \boxed{v_i(w^*) + \sum_{j \neq i} b_j(w^*)} - \underbrace{\max_{w} \sum_{j \neq i} b_j(w)}_{\text{not depends on } b_i}$$

social welfare of allocation $w^*$

$$w^* = \arg\max_{w} \sum_{j=1}^{n} b_j(w)$$

objective of Agent $i$ is the same as the objective of designer.