Cake cutting under conflicting constraints

Ayumi Igarashi, 22nd June 2022

1. Hadi Hosseini, Ayumi Igarashi, Andrew Searns,
Fair division of time: multi-layered cake cutting, IJCAI 2020.

2. Ayumi Igarashi and Frédéric Meunier,
Envy-free division of multi-layered cakes, WINE 2021

How to cut a cake fairly?

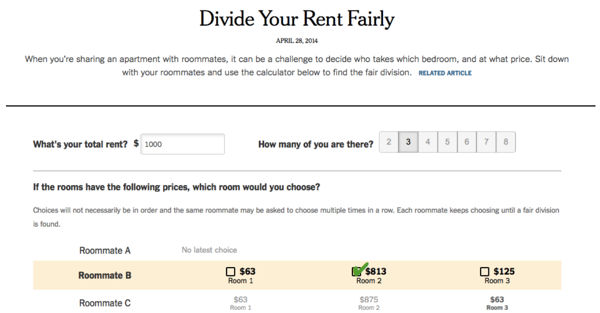
- Cut and Choose protocol [Bible]

- Formulated as a mathematical problem [Hugo Steinhaus, 1948]
- Existence of an envy-free division for n agents [Dubins-Spanier, 1961]
- Bounded protocol [Aziz-Mackenzie, FOCS 2016]

Cake cutting

- The goal is to fairly distribute divisible resource among agents.
- Applications: division of time, task, and etc.

New York Times
Rent Division Calculator



- I am going to talk about cake cutting under conflicting constraints.

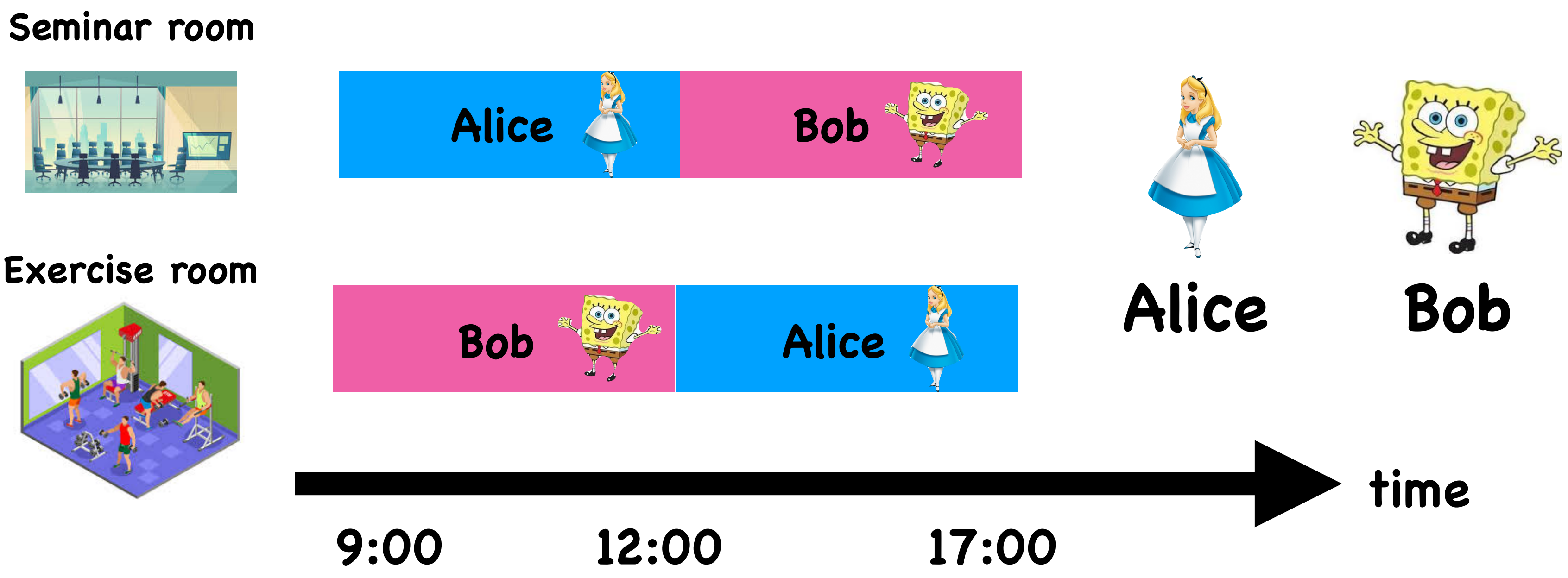
  1. Hadi Hosseini, Ayumi Igarashi, Andrew Searns, IJCAI 2020.

  2. Ayumi Igarashi and Frédéric Meunier, WINE 2021

- Fixed point theorem and their generalizations have found many applications in game theory. → It is also the case for cake-cutting.

Motivation

Alice and Bob wish to use multiple college facilities over different periods of time.



How do we allocate different time intervals?

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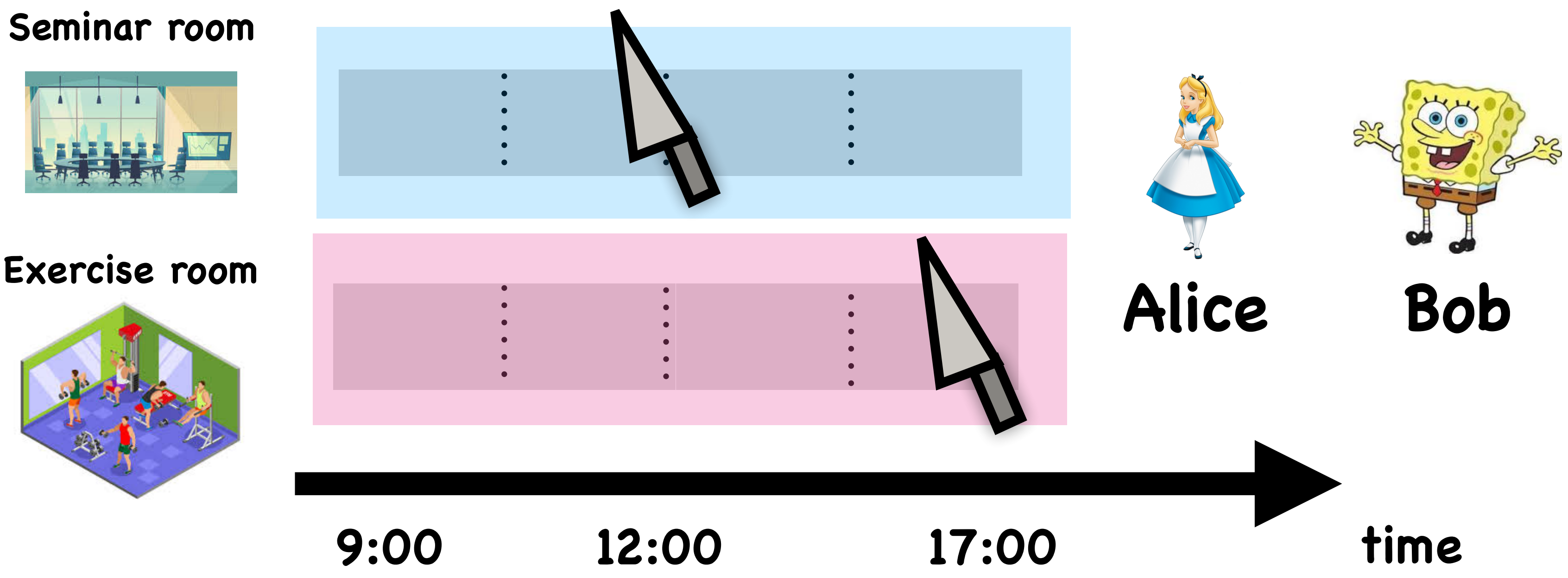

- Naive approach: treat each interval independently?

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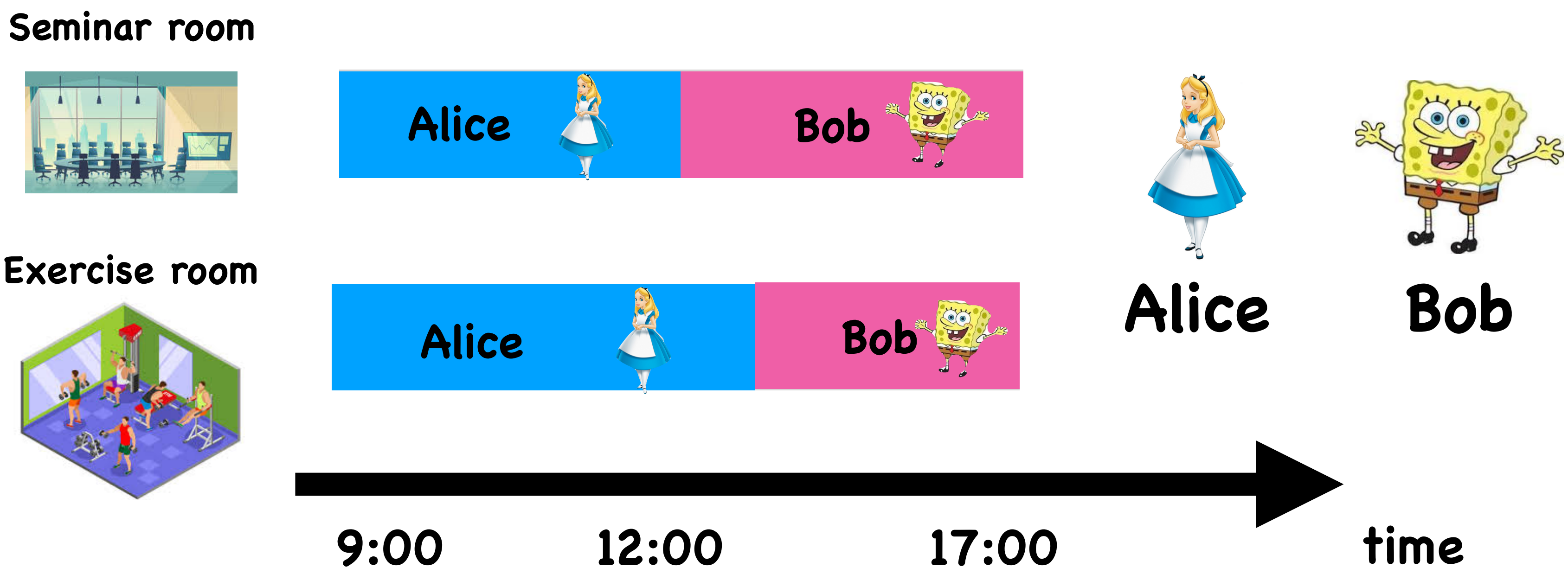


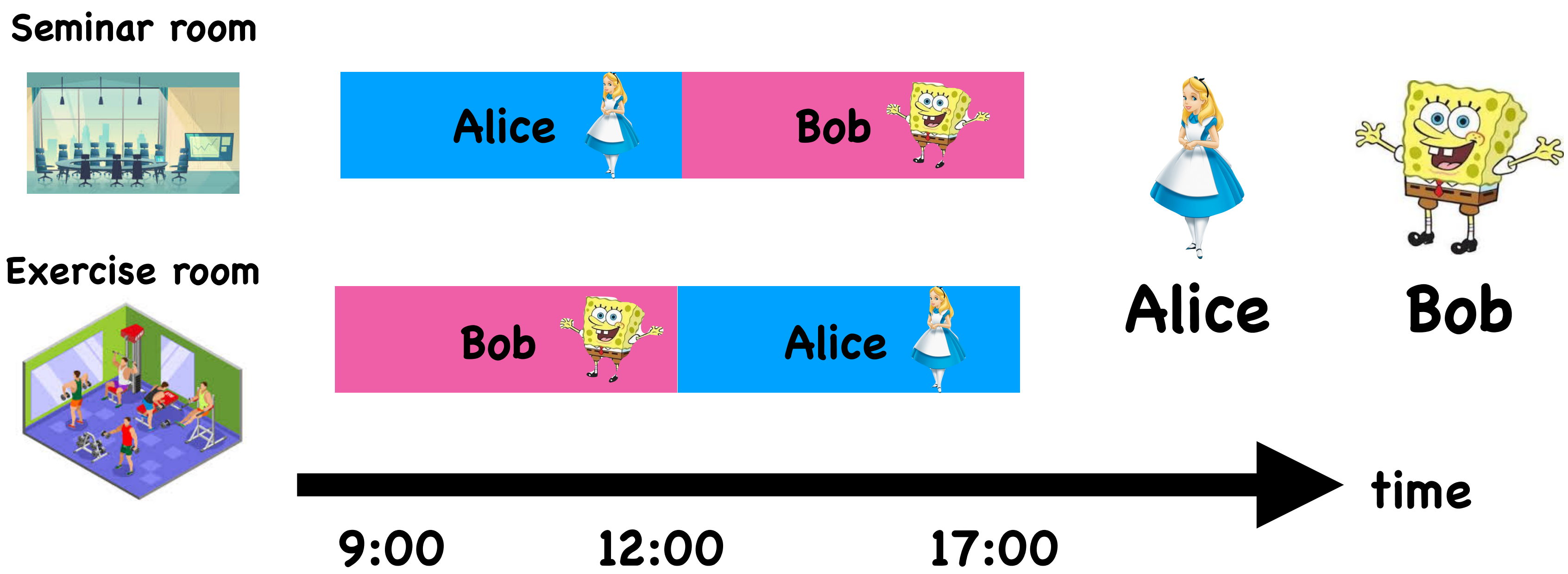
- Naive approach: treat each interval independently?

→ Agents can only use a single facility at a time.

Hosseini-Igarashi-Searns (IJCAI 2020)

Initiate the study of multi-layered cake cutting.

Each layer represents a divisible resource.



Feasibility: each agent's share must be non-overlapping.

Contiguity: each agent's share must be contiguous for each layer.

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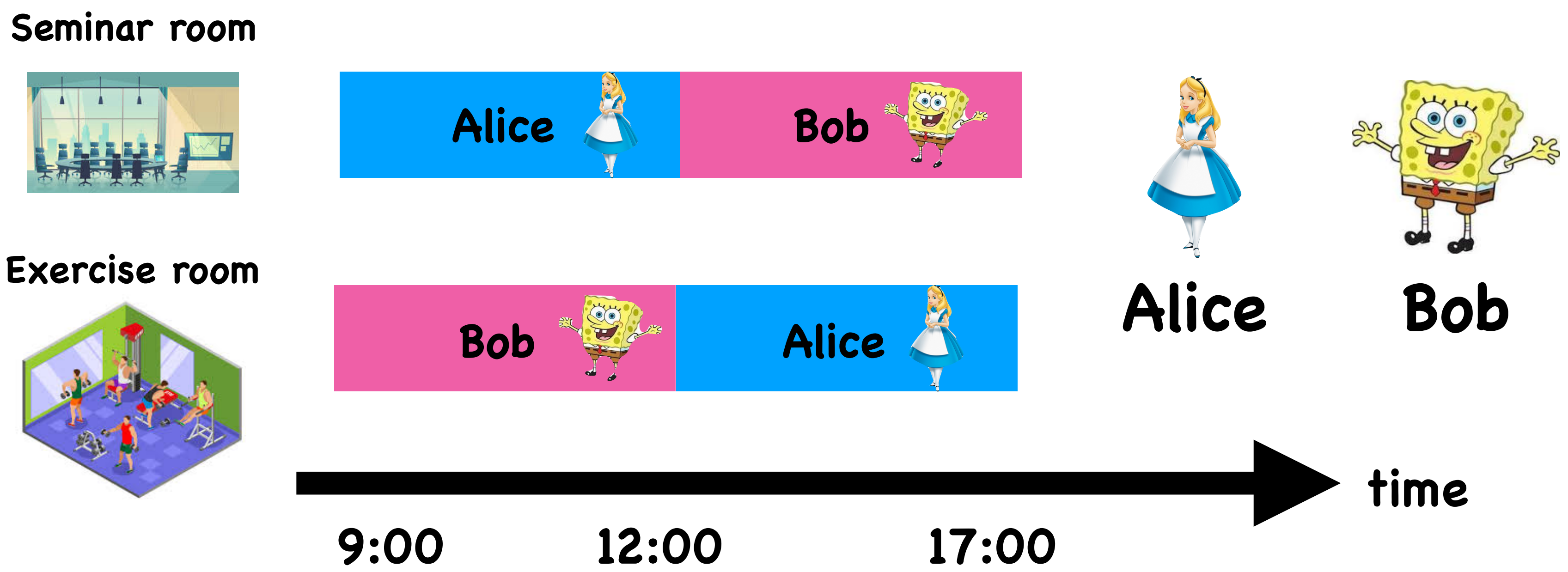

What fairness guarantees can be achieved under feasibility and contiguous constraints?

Igarashi-Meunier (WINE 2021)

It is open whether an envy-free multi-division that is contiguous and feasible exists for the general case (even when the number of layers m=2 and the number of agents n=3).

- Envy-free division that is contiguous and feasible exists when $m \leq n$ and n is a prime power.

  The existence holds even for non-monotone preferences!

- FPTAS for finding an envy-free division that is contiguous and feasible when m=2 and n=3.

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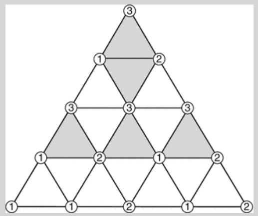
- FPTAS

  contigu

- The standard proof using Sperner's lemma (a combinatorial analogue of Brower's fixed point) may not work here.

- Instead of Sperner, we use a more general fixed point theorem proven by Volovikov (1996).



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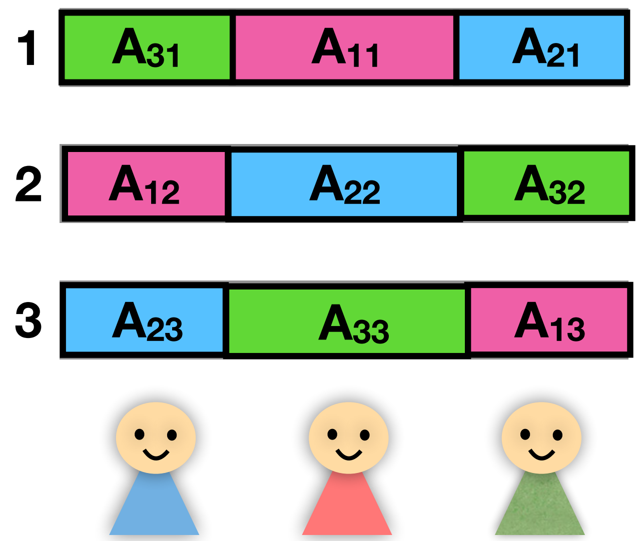
  For a non-prime number, a counter example exists under the choice function model [Avvakumov and Karasev 2020; Panina and Zivaljevic 2021]

Model



- n agents
- A layered cake: m intervals $[0,1]$
- Multi-division $\mathscr{A} = (\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n)$

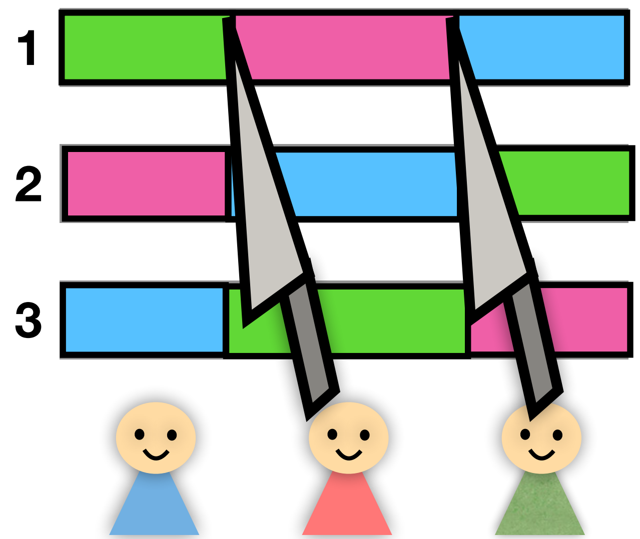
  $\mathscr{A}_i = (A_{ij})_{j=1,\ldots,m}$

  Contiguity: each A_{ij} is contiguous.

  Feasibility: no two pieces of distinct layers overlap.

  $A_{ij} \cap A_{ij'} = \emptyset \forall j \neq j'$

Model



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- Short knife moves over one layer.

  Long knife moves over the whole cake.

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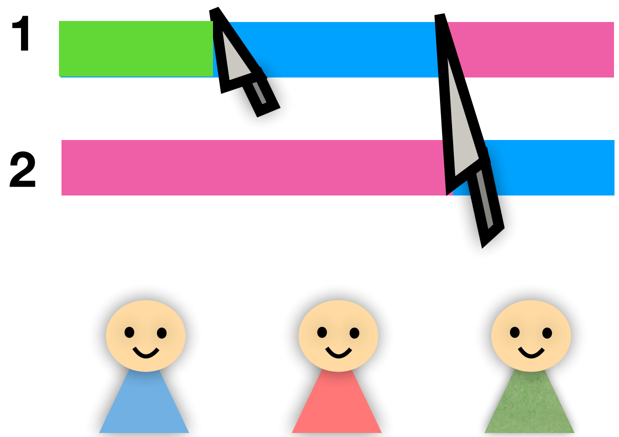

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Preferences

- Each agent i has a choice function c_i that given a multi-division, returns the set of favorite bundles.
  - Closed preferences

    If an agent prefers the i-th bundle in the converging sequence of multi-divisions, then she prefers the i-th bundle in the limit.

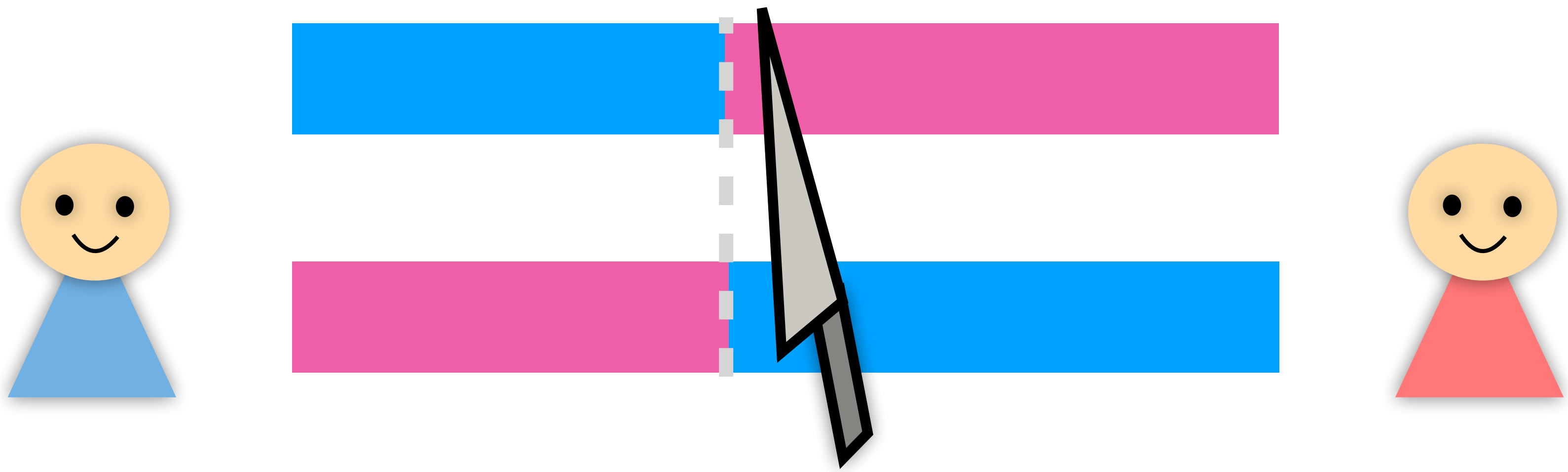
  - Monotone preferences

    Agent weakly prefers a layered piece $\mathscr{L}$ to another layered piece $\mathscr{L}'$ whenever $\mathscr{L}' \subseteq \mathscr{L}$

Envy-freeness

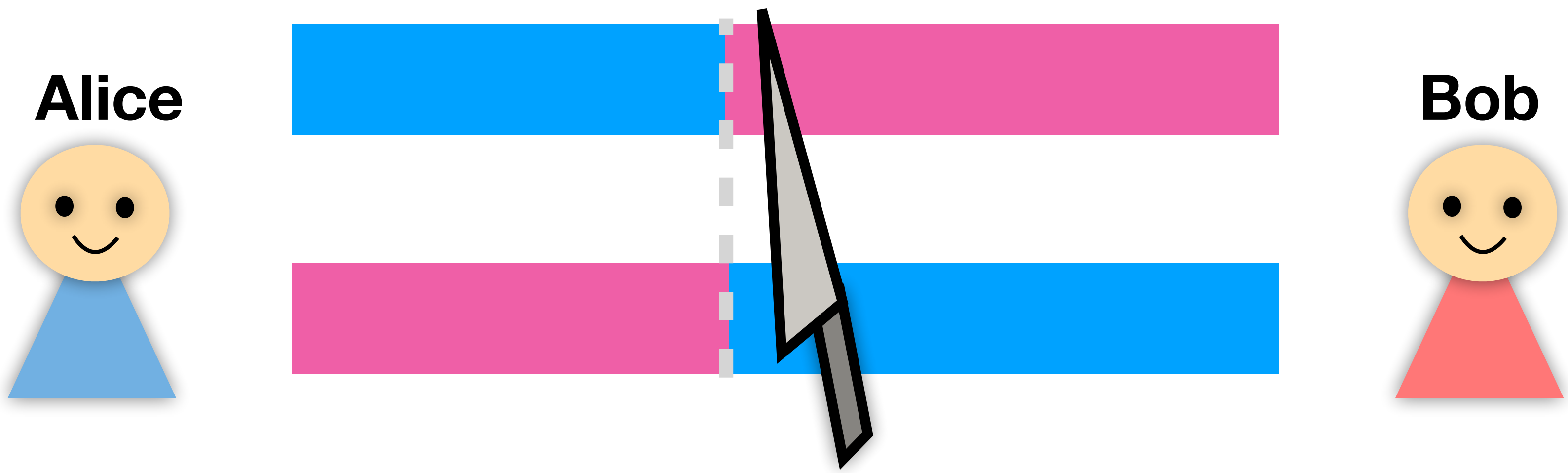
- Envy-freeness: no agent envies others.

$$\mathscr{A}_i \in c_i(\mathscr{A}) \qquad \forall i = 1,2,\ldots,n$$



Cut and Choose for two agents and two layers

- "Diagonal bundles" over two layers.



1. Alice cuts the cake into two equally valued diagonal bundles.
2. Bob chooses a preferred bundle.

More than two agents?

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Igarashi and Meunier (WINE 2021)

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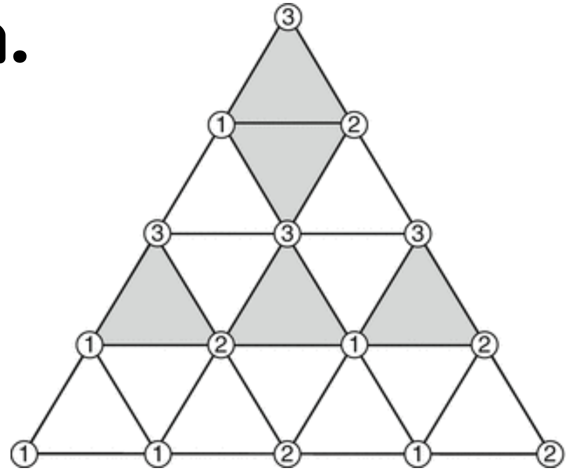
- For a non-prime number, a counter example exists under the choice function model [Avvakumov and Karasev 2020; Panina and Zivaljevic 2021]

The standard case

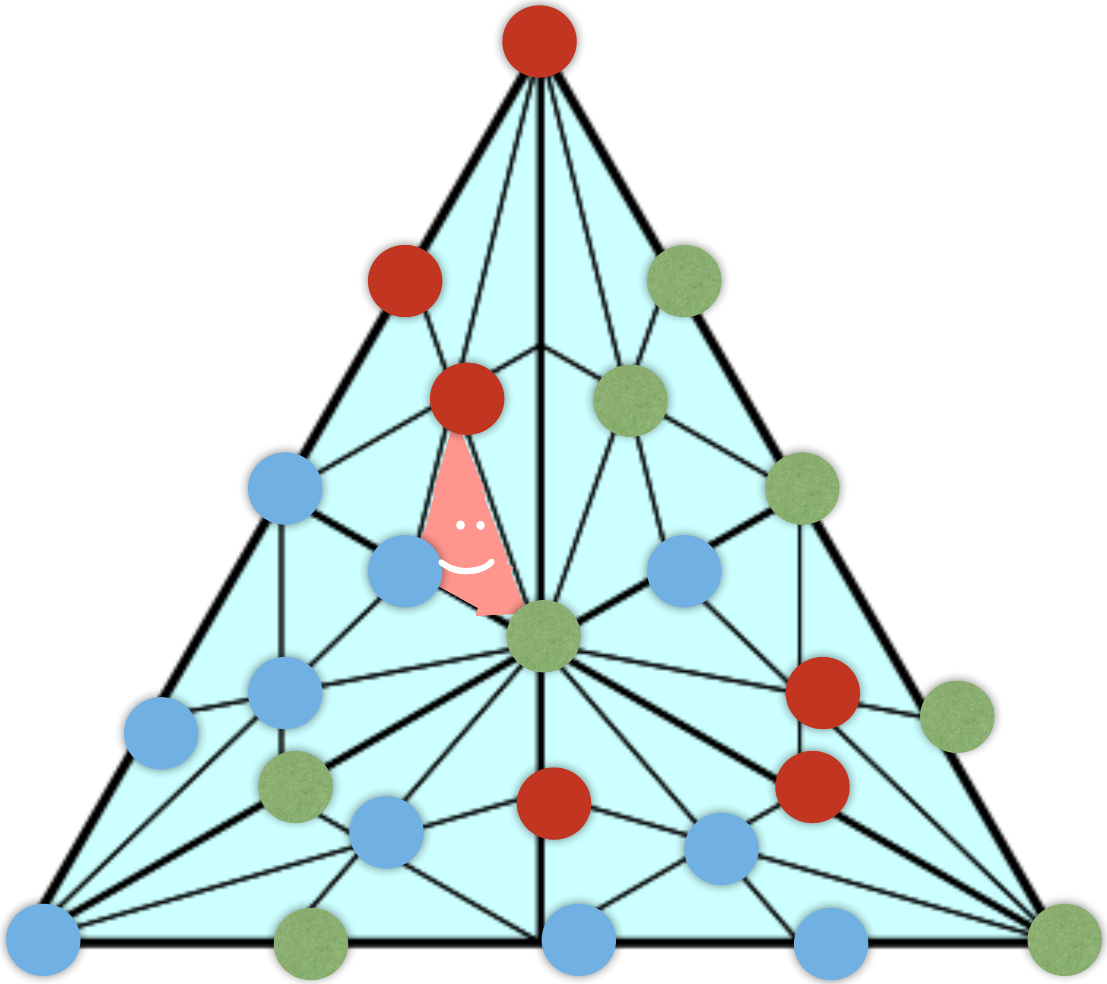
- How do we prove the existence of EF division for the standard one-layered cake?

  → Use Sperner's lemma [Su 1999]

Combinatorial version of Brouwer's fixed point theorem.



Sperner's lemma

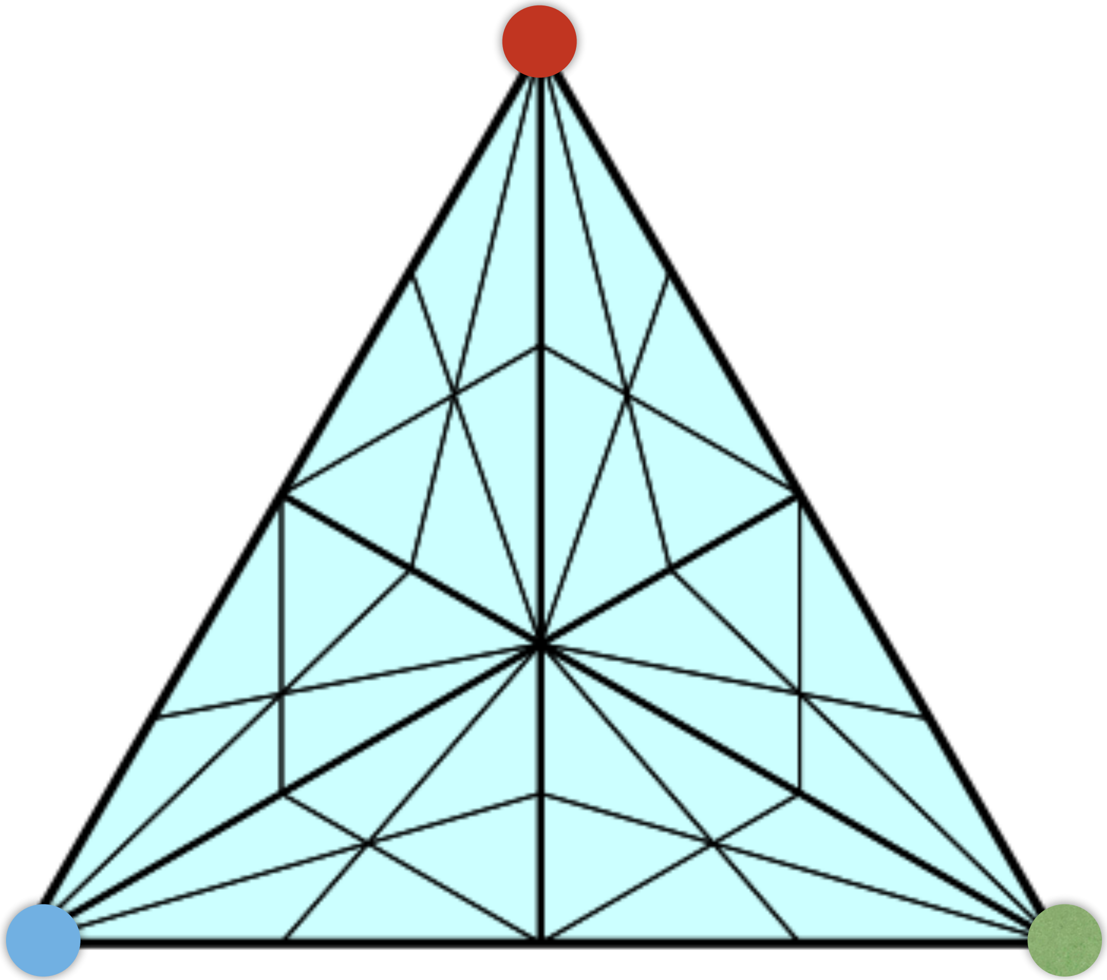


Sperner's lemma

- Every corner vertex has a distinct color.
- The vertices along any edge of the big triangle have only two colors, the two colors at the endpoints of the edge.

→ A fully colored triangle exists.

Sperner's lemma



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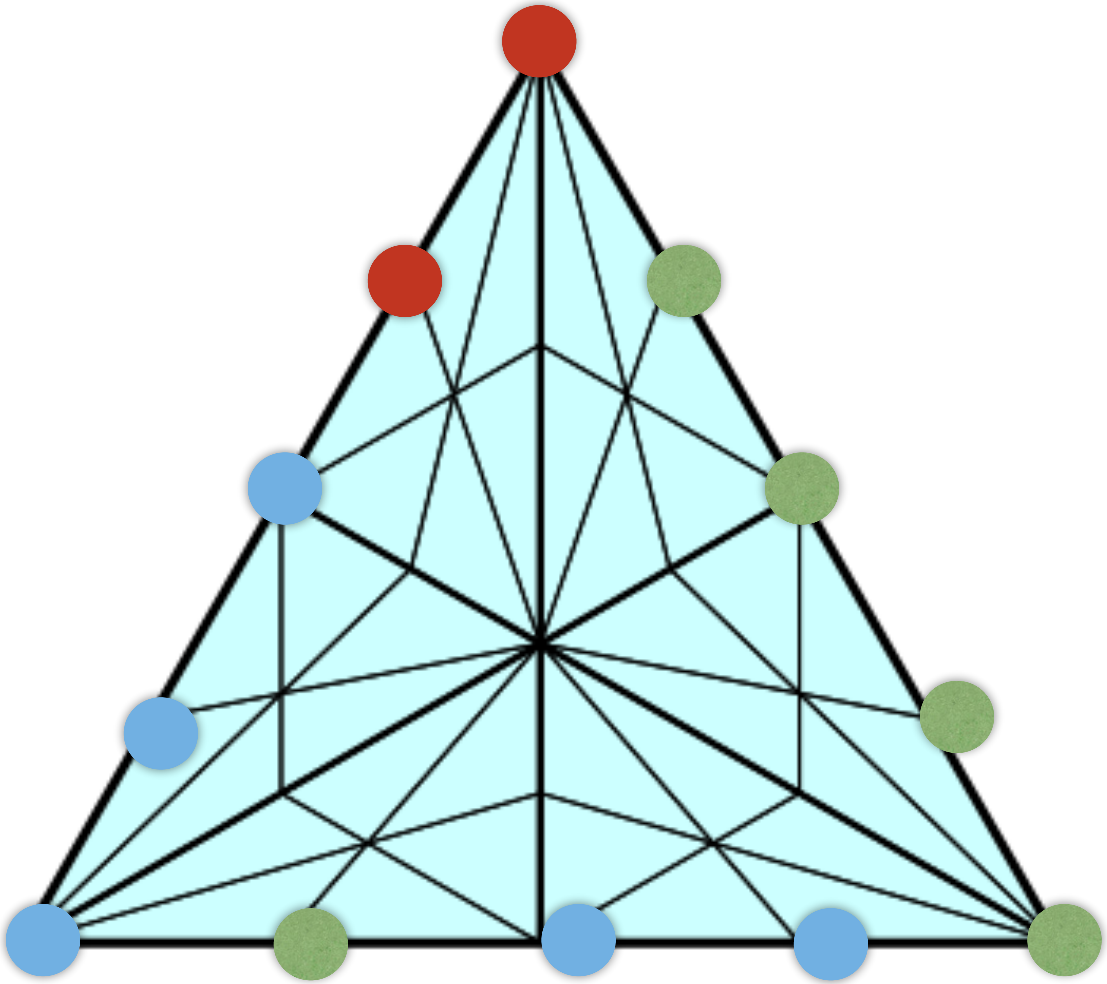

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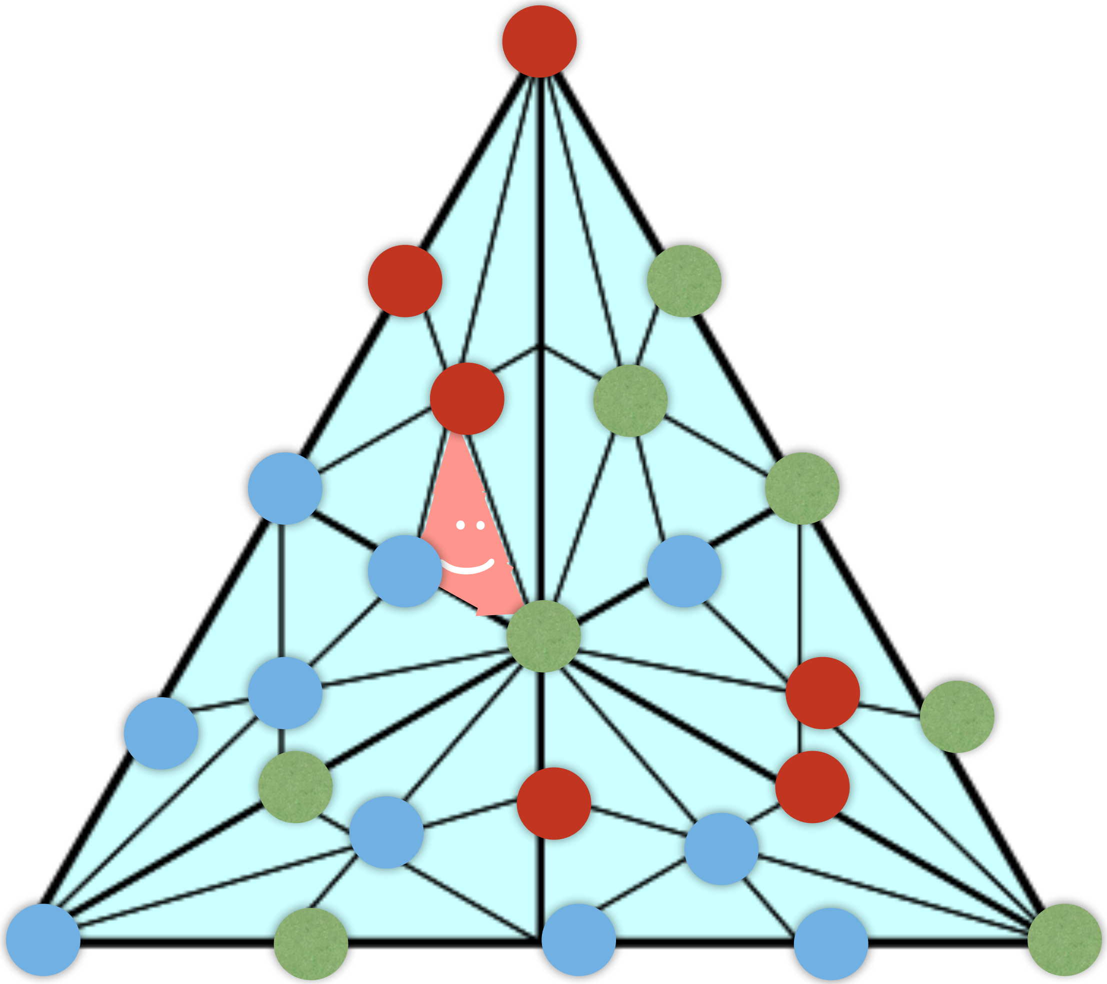

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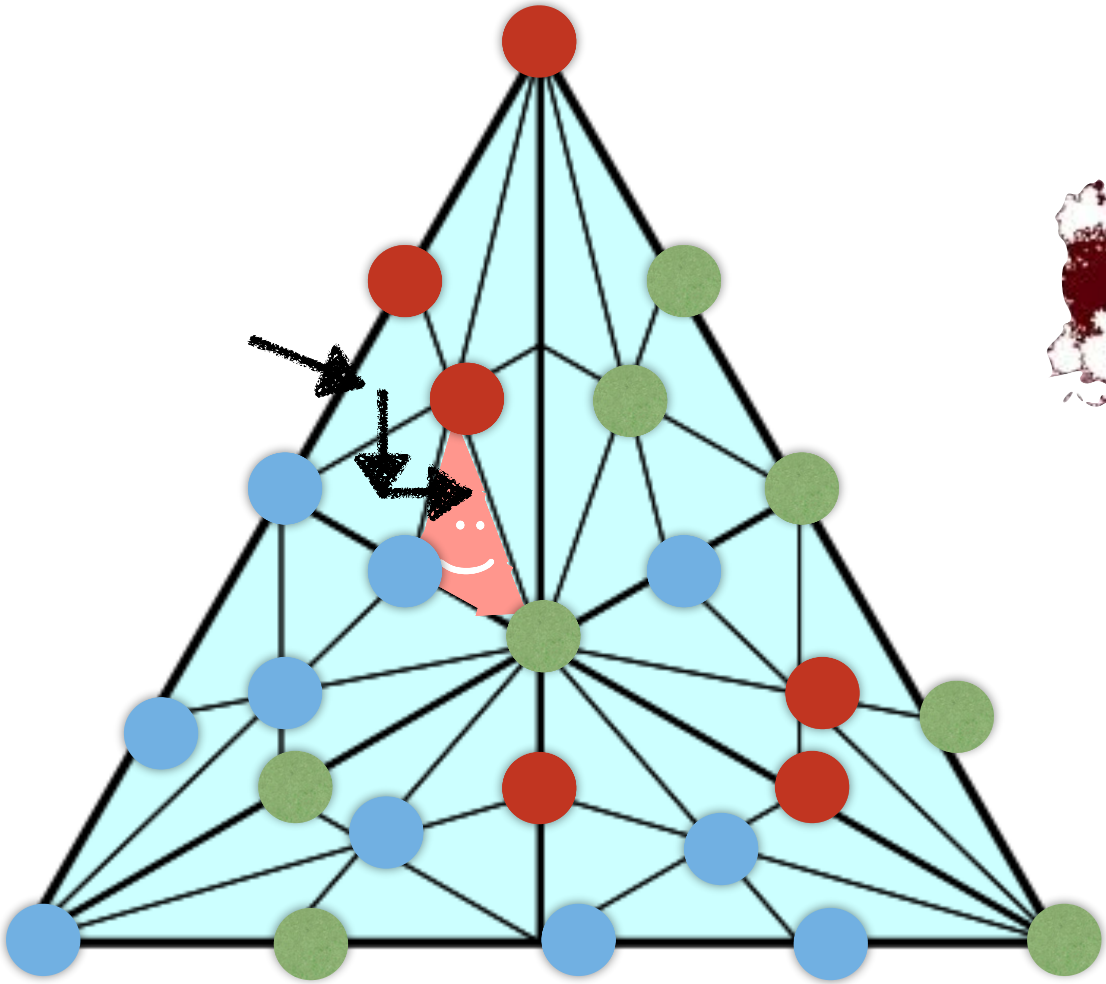
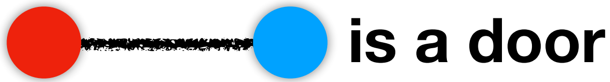
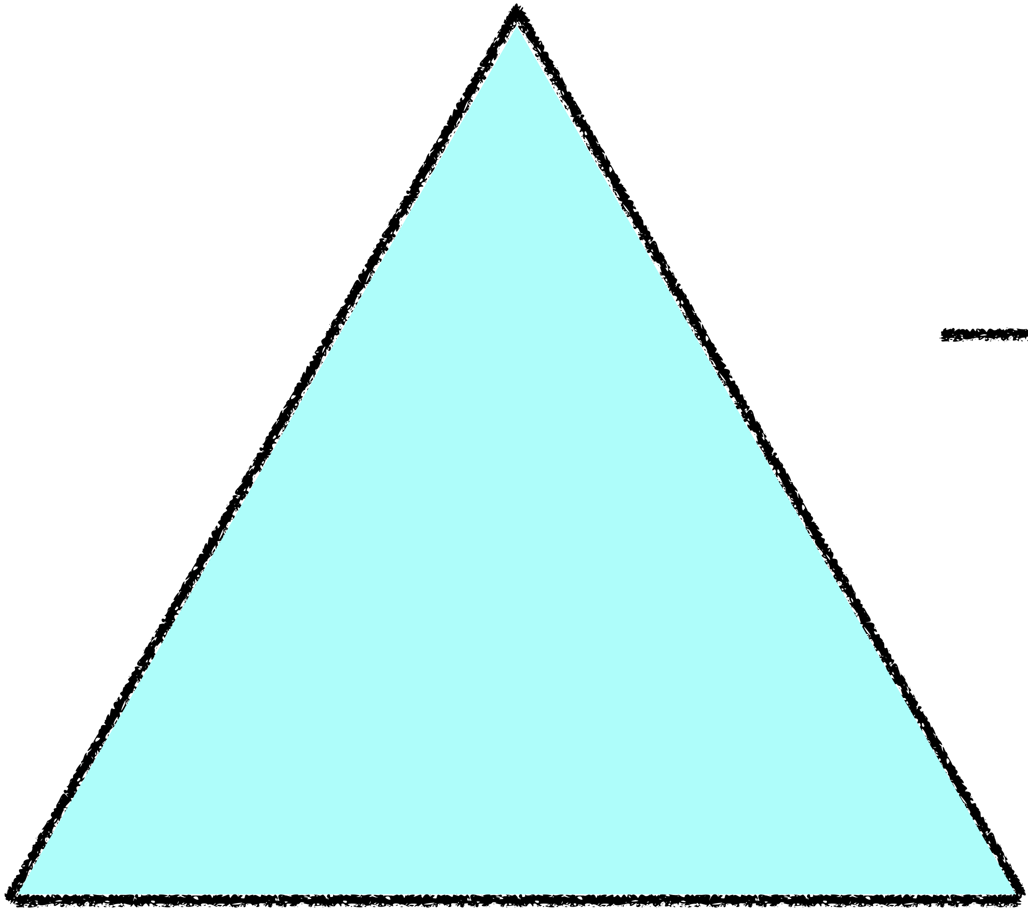


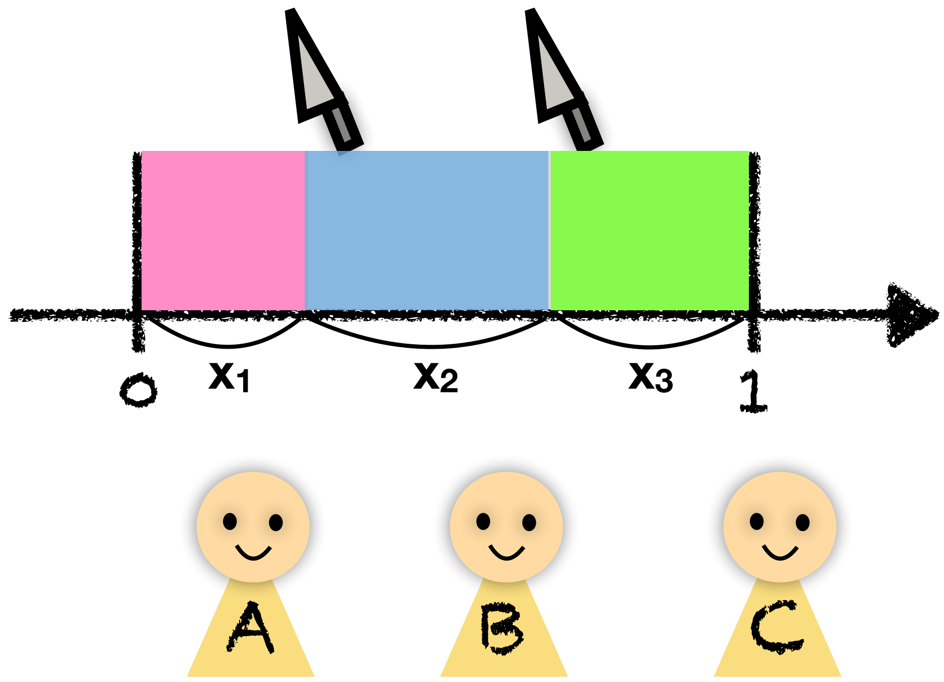


Sperner's Lemma and cake-cutting

Encode the divisions by the points of the standard simplex.

$$x_1 + x_2 + x_3 = 1$$





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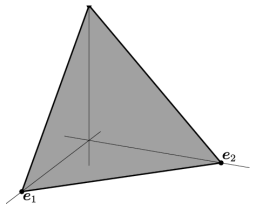
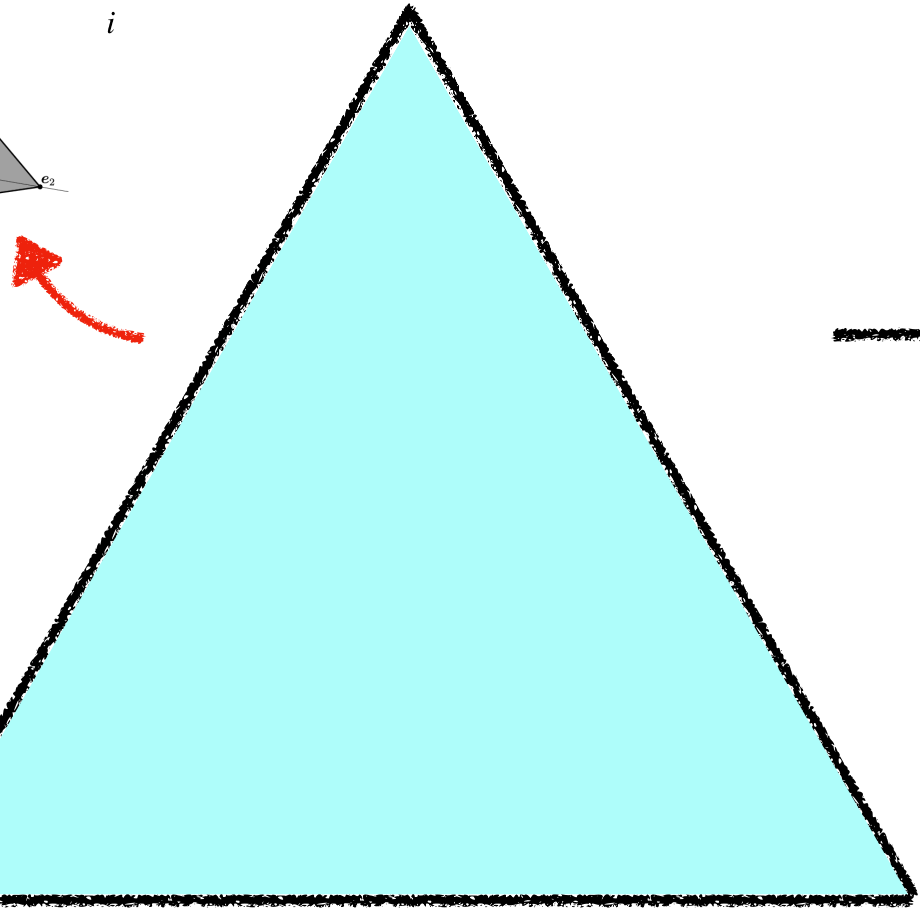
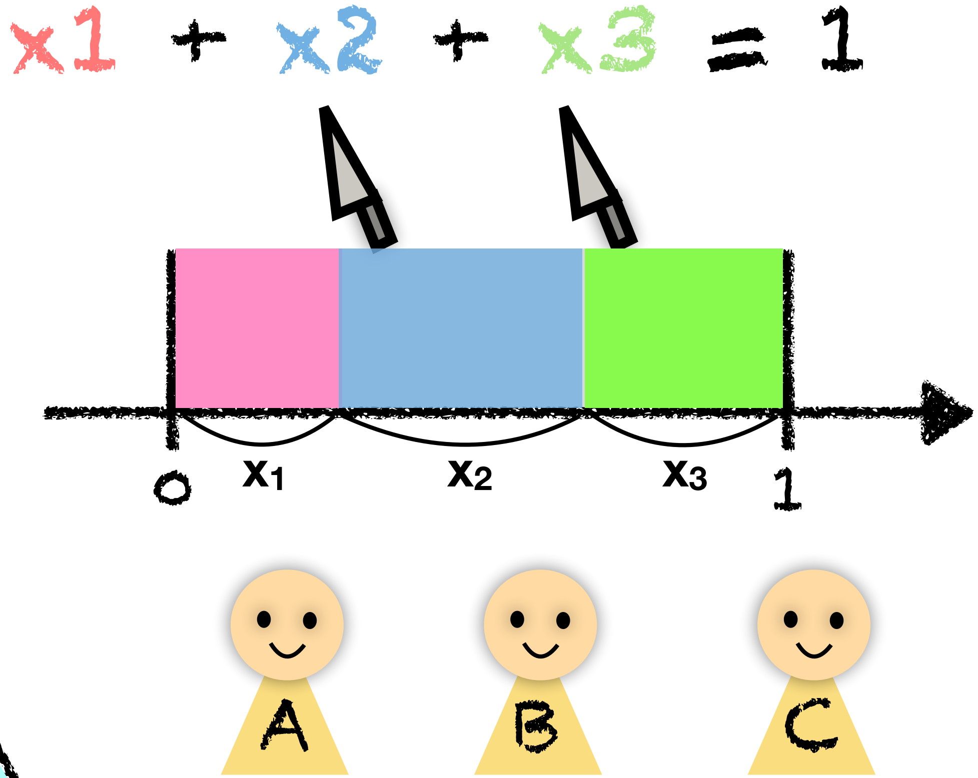
$$\Delta^2 = \{ x \in \mathbb{R}^3_+ \mid \sum_i x_i = 1 \}$$



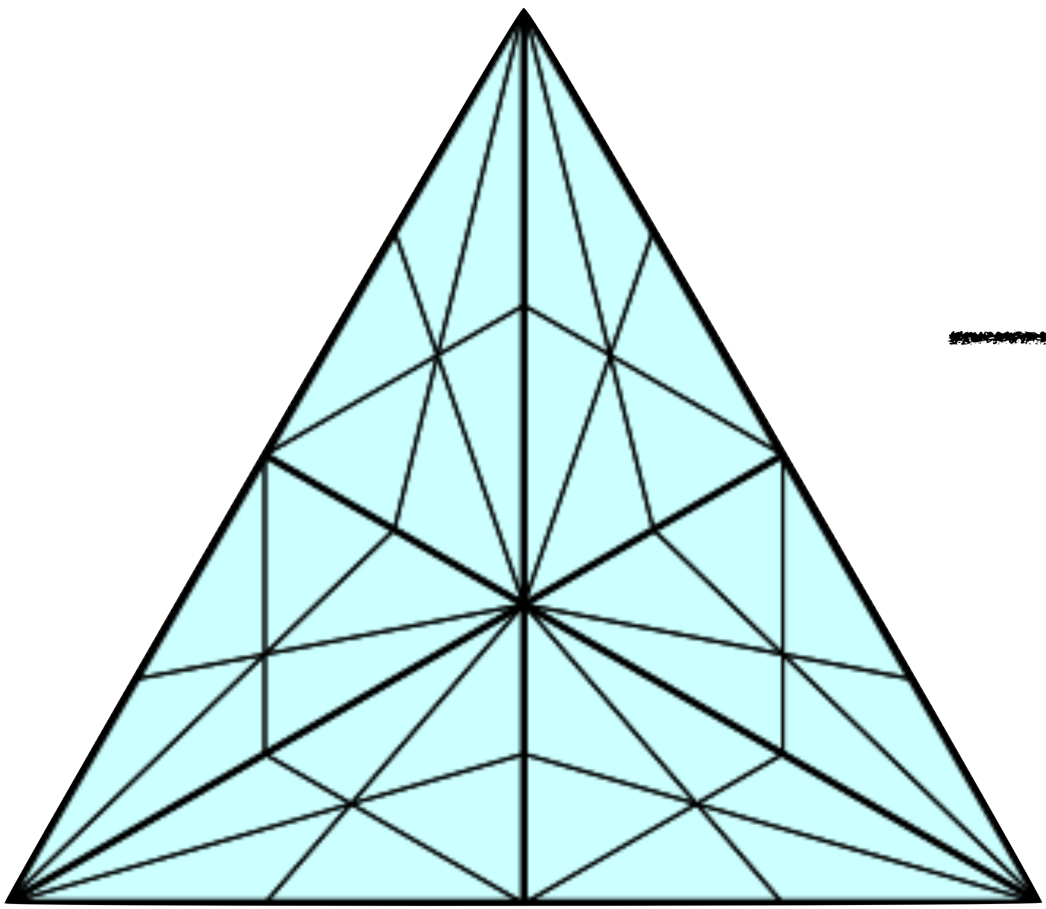


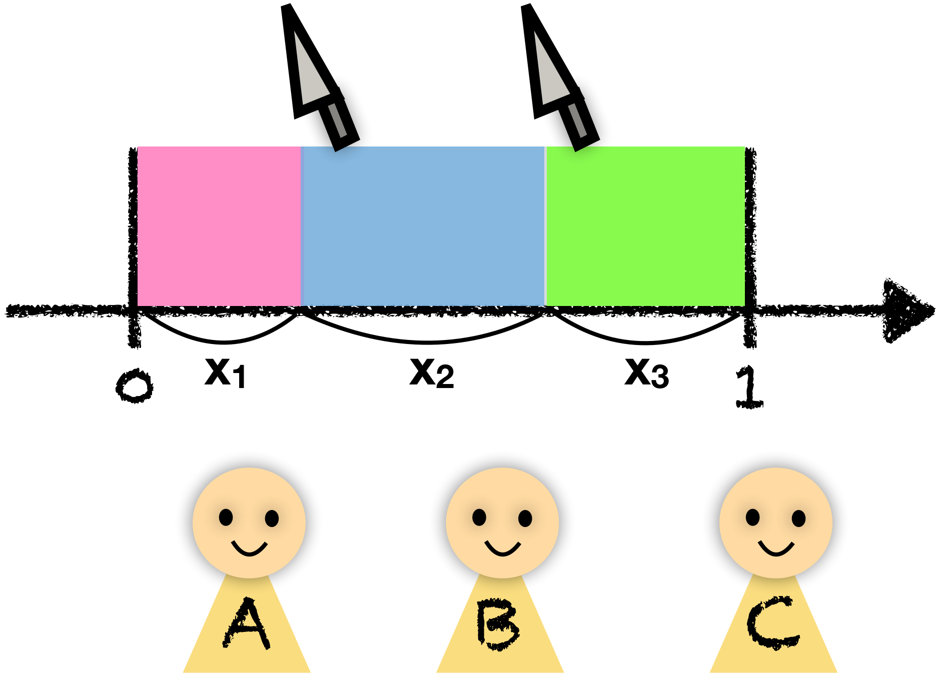


Sperner's Lemma and cake-cutting

1. Triangulate the simplex and assign owner label so that every small triangle receives different agent labels.

$x1 + x2 + x3 = 1$





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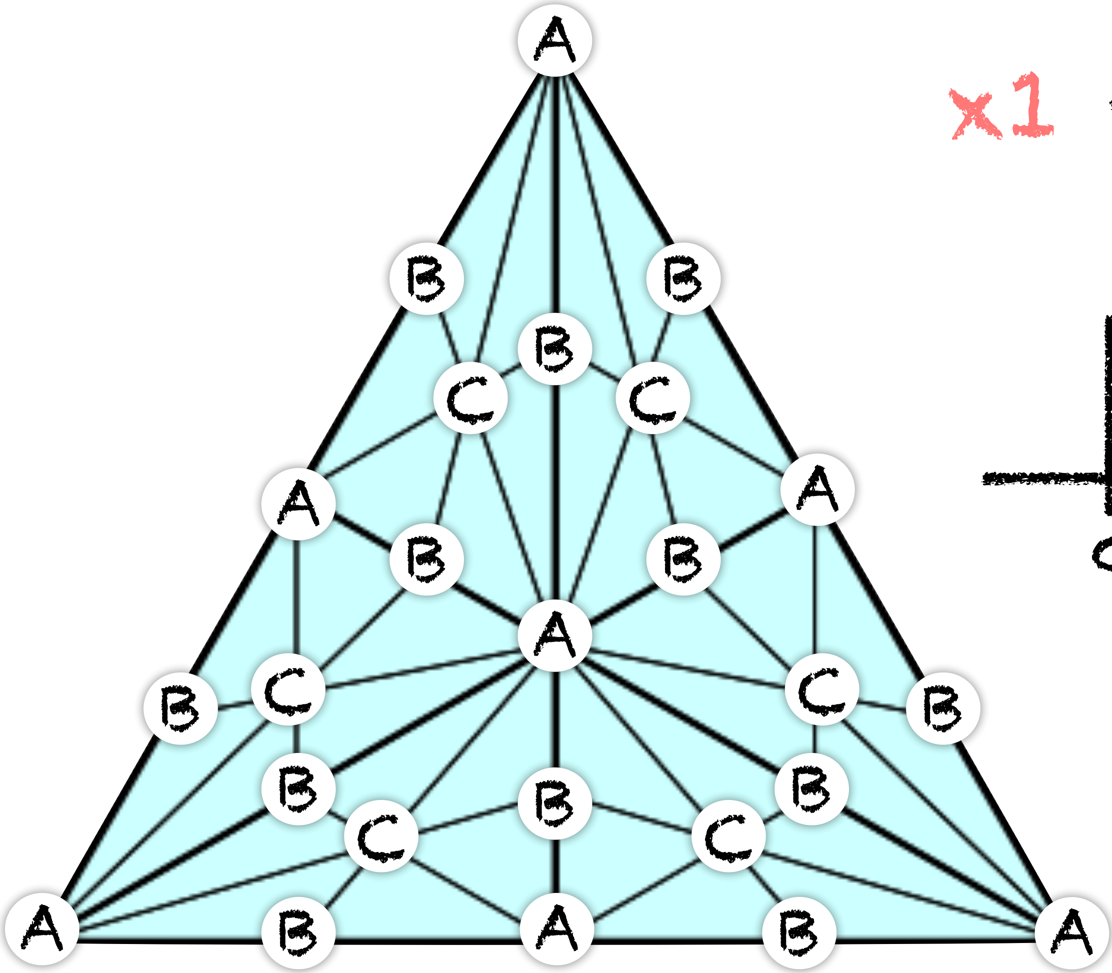
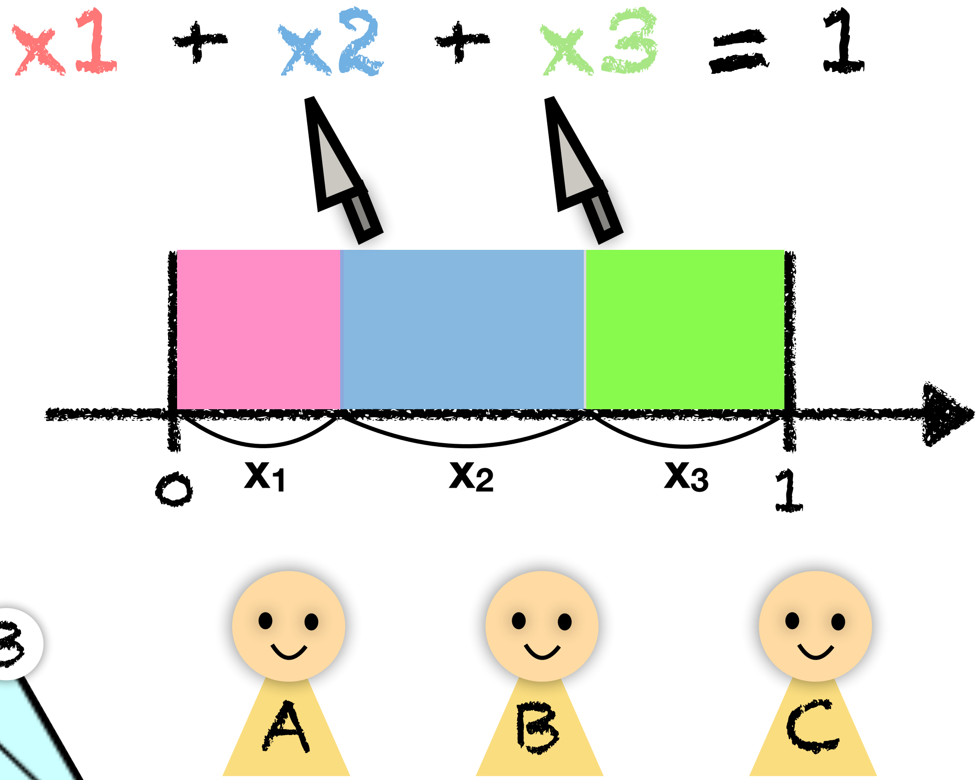
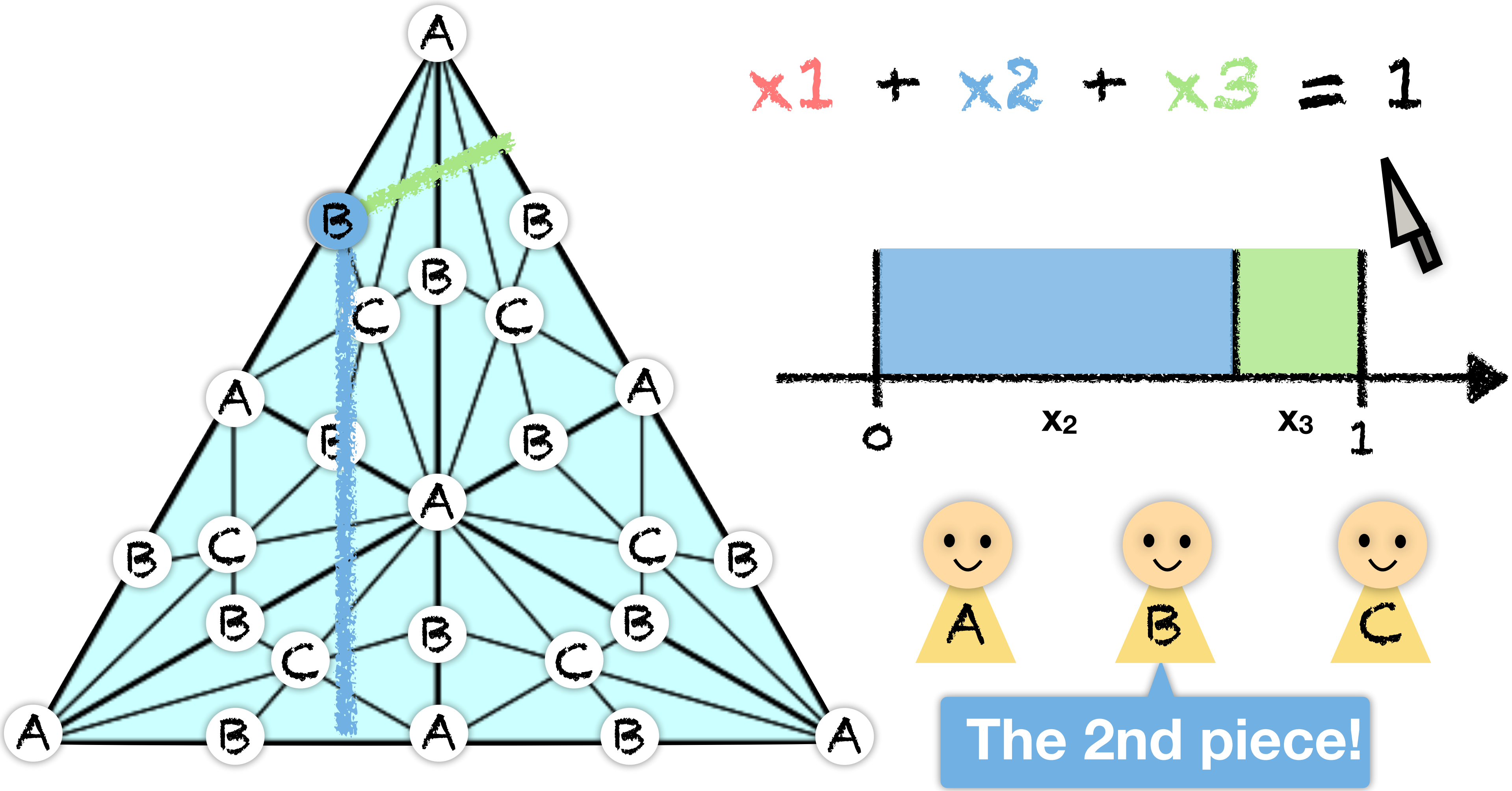




Sperner's Lemma and cake-cutting

2. Each owner assigns the color of the favorite piece.



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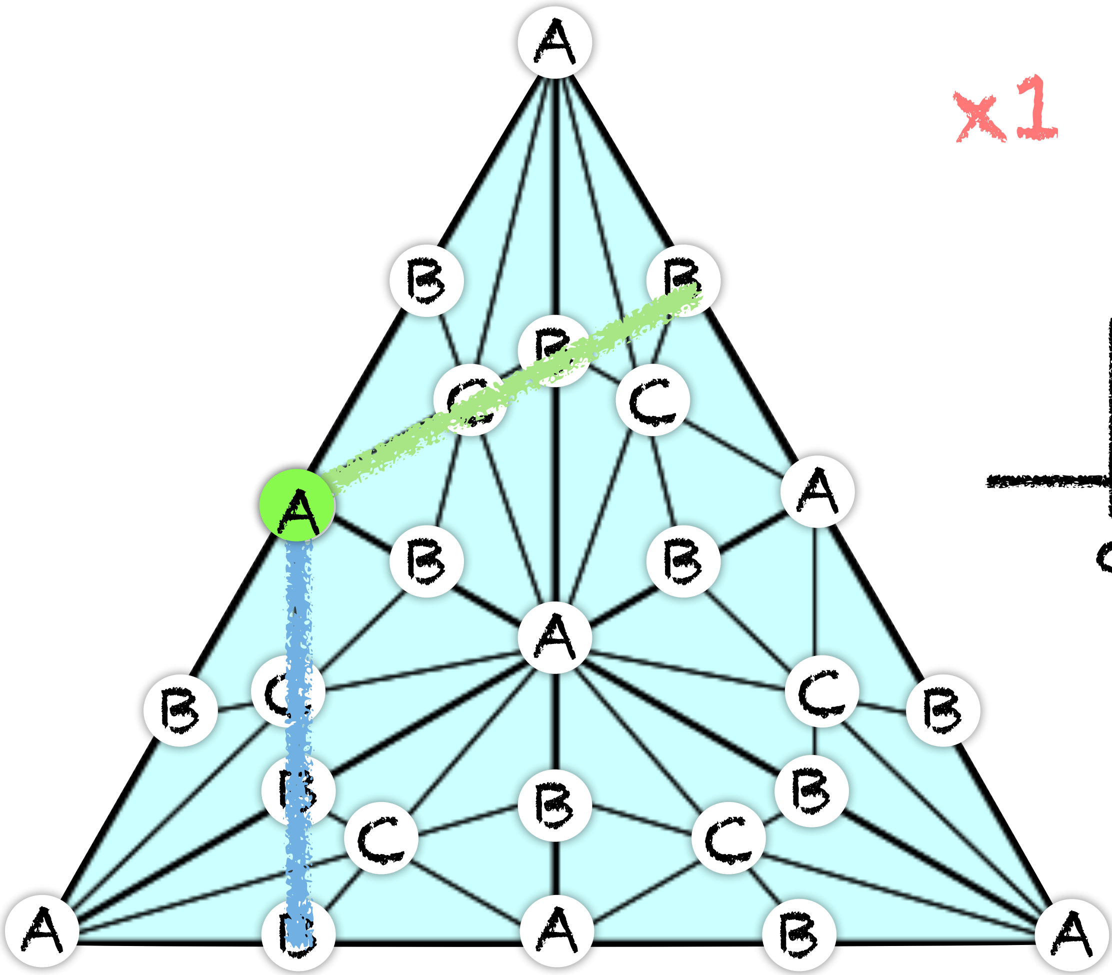

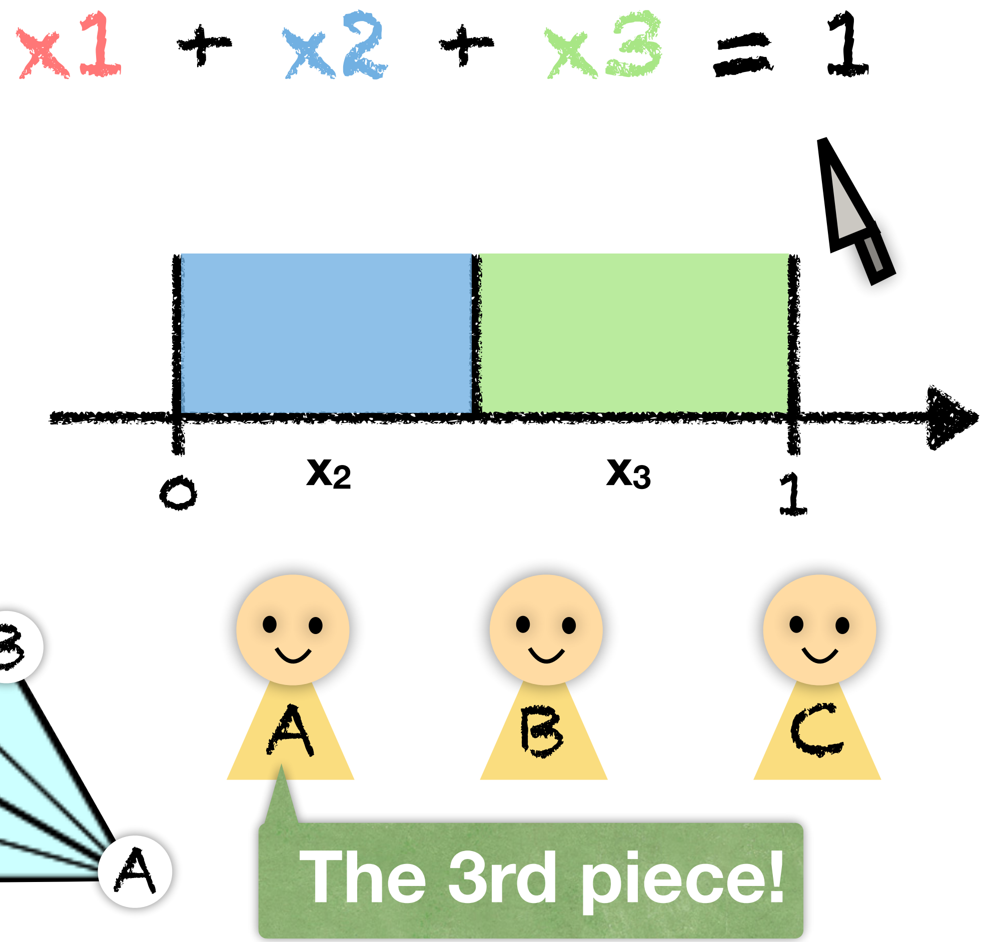

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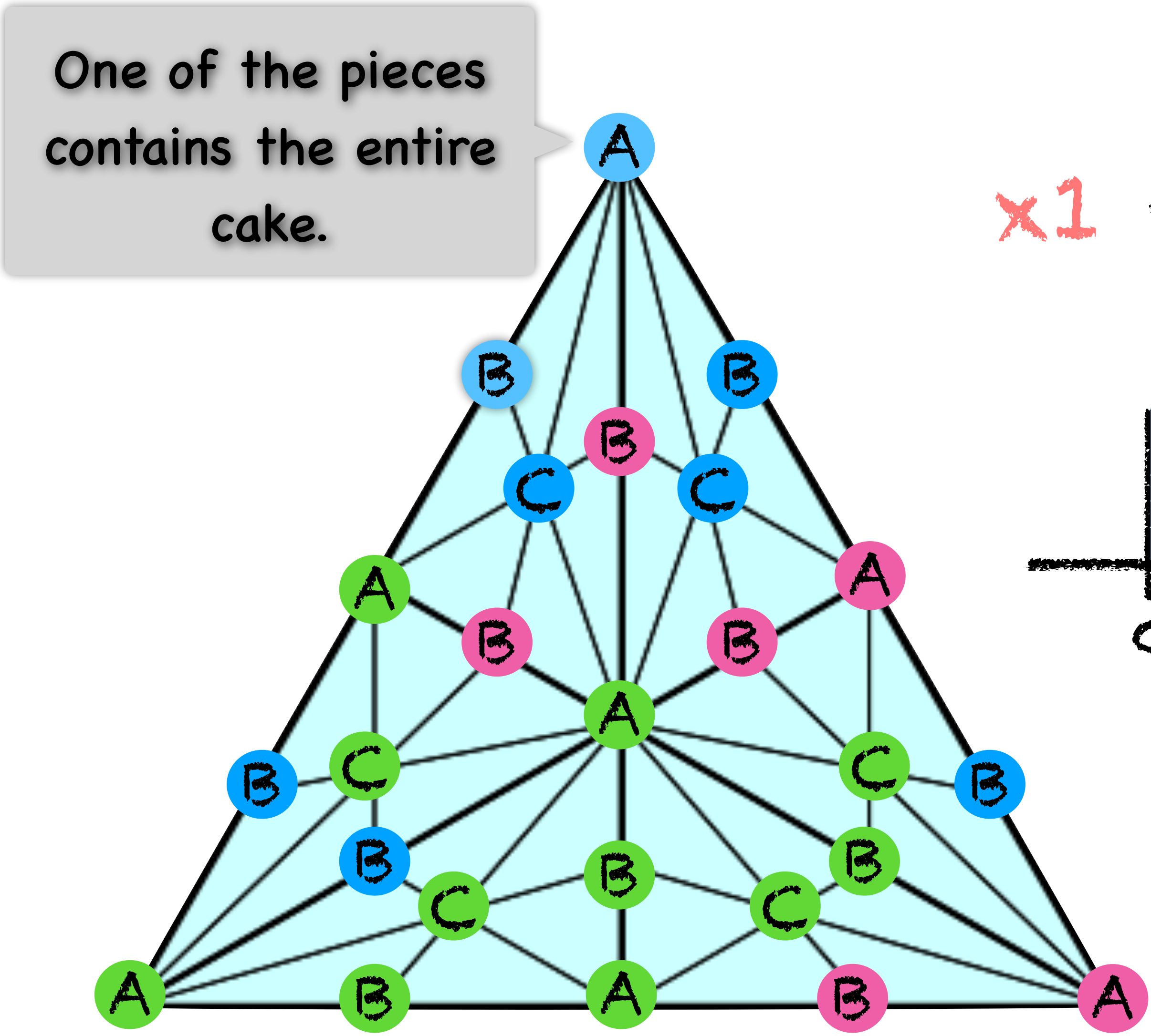

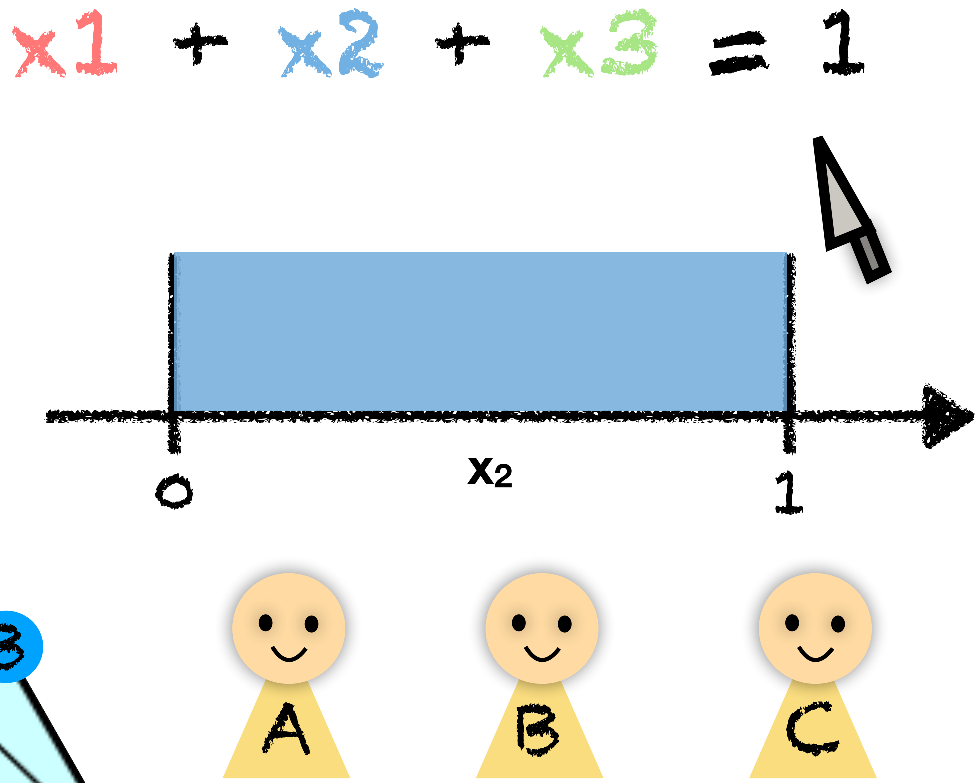

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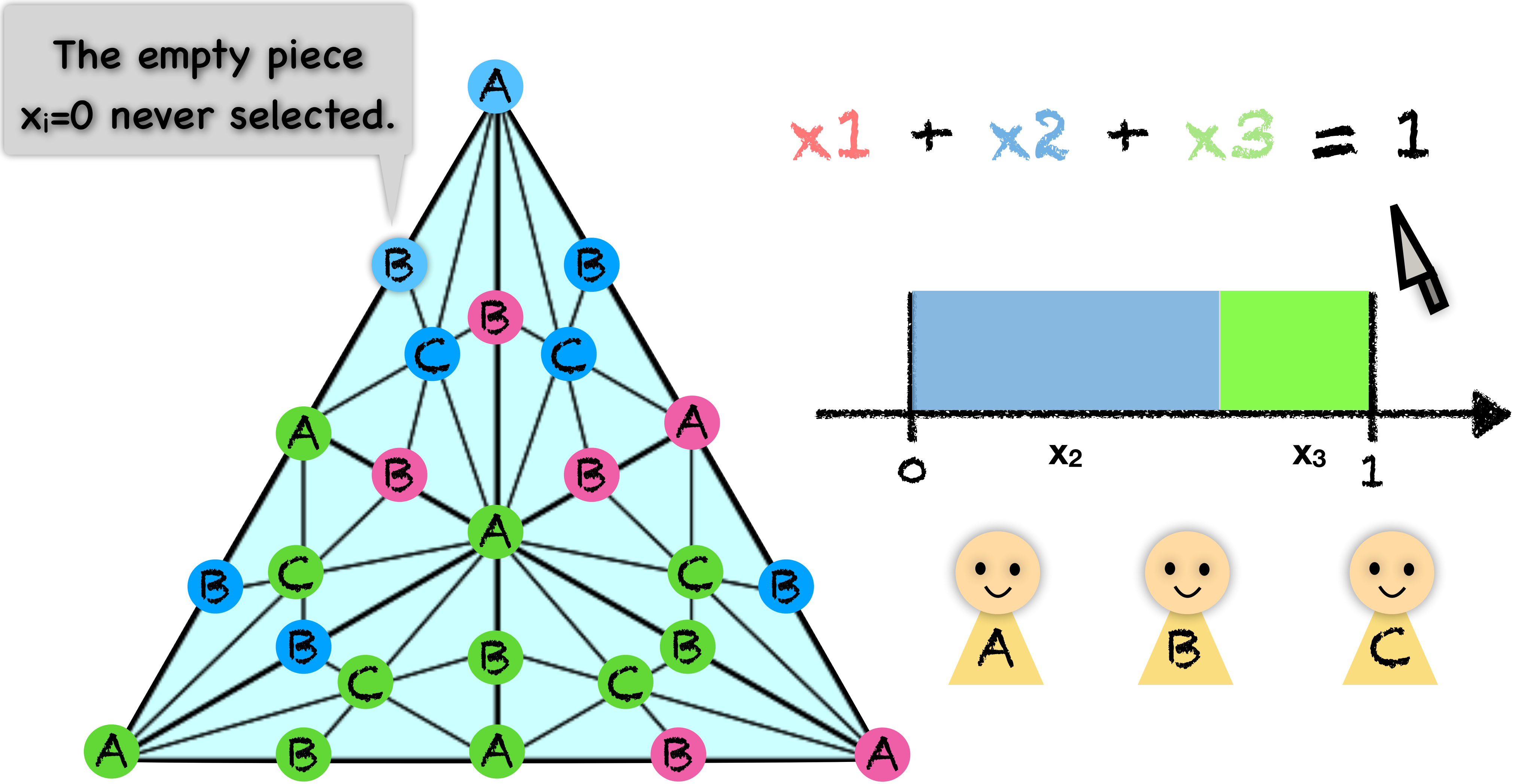
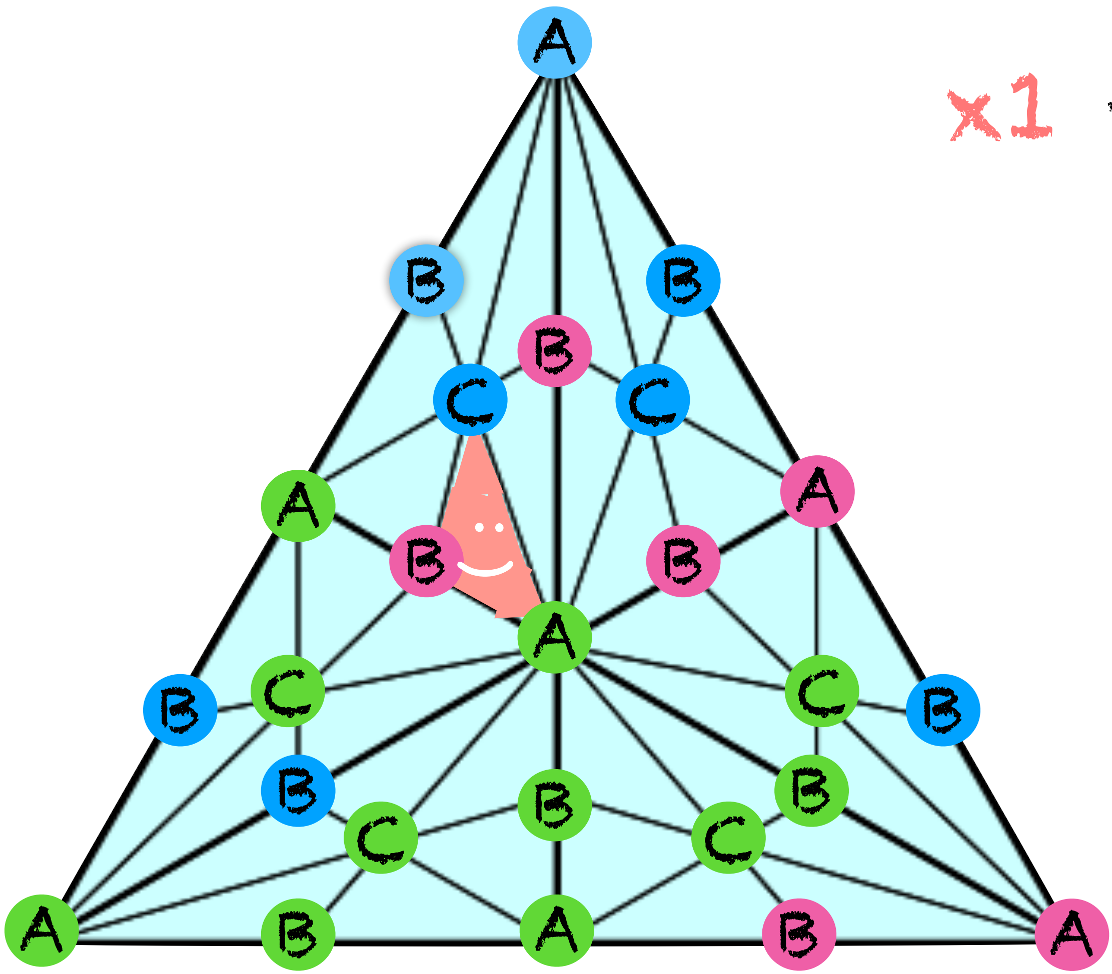


Sperner's Lemma and cake-cutting

3. Apply Sperner's lemma and get a fully colored triangle.



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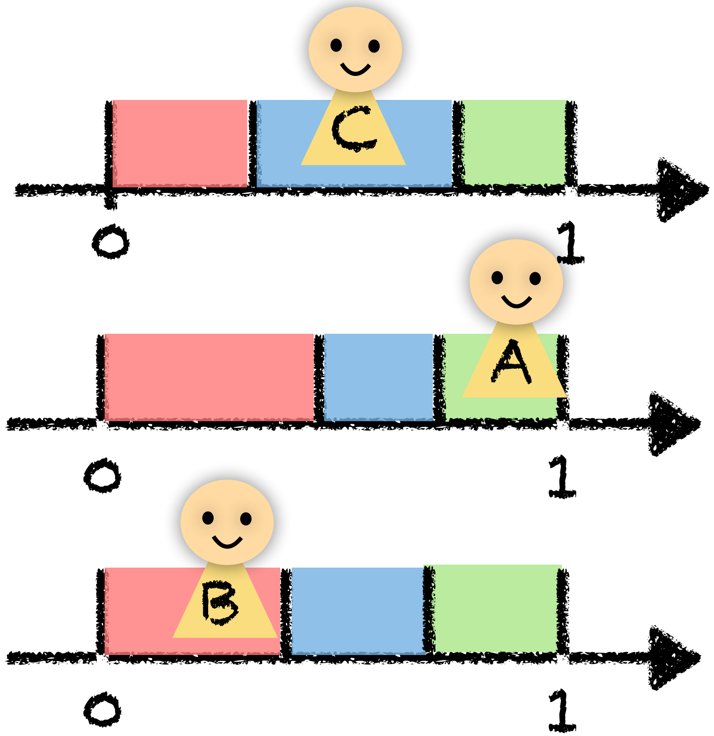
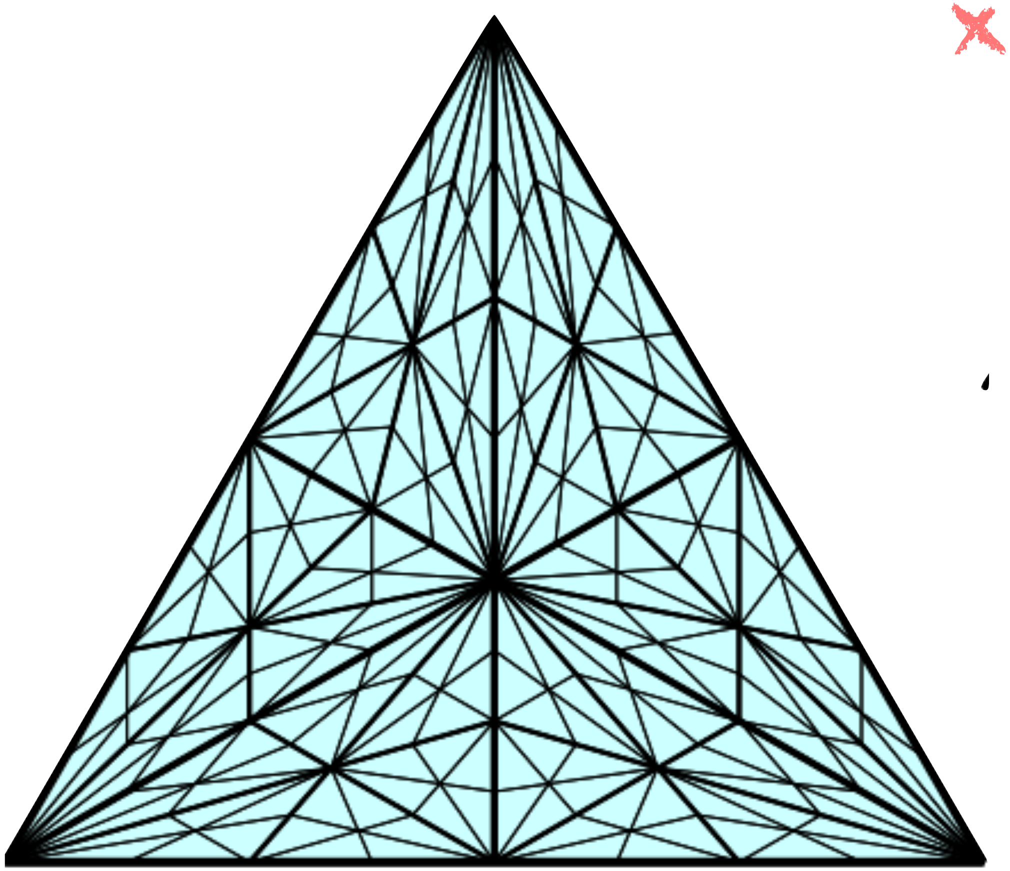
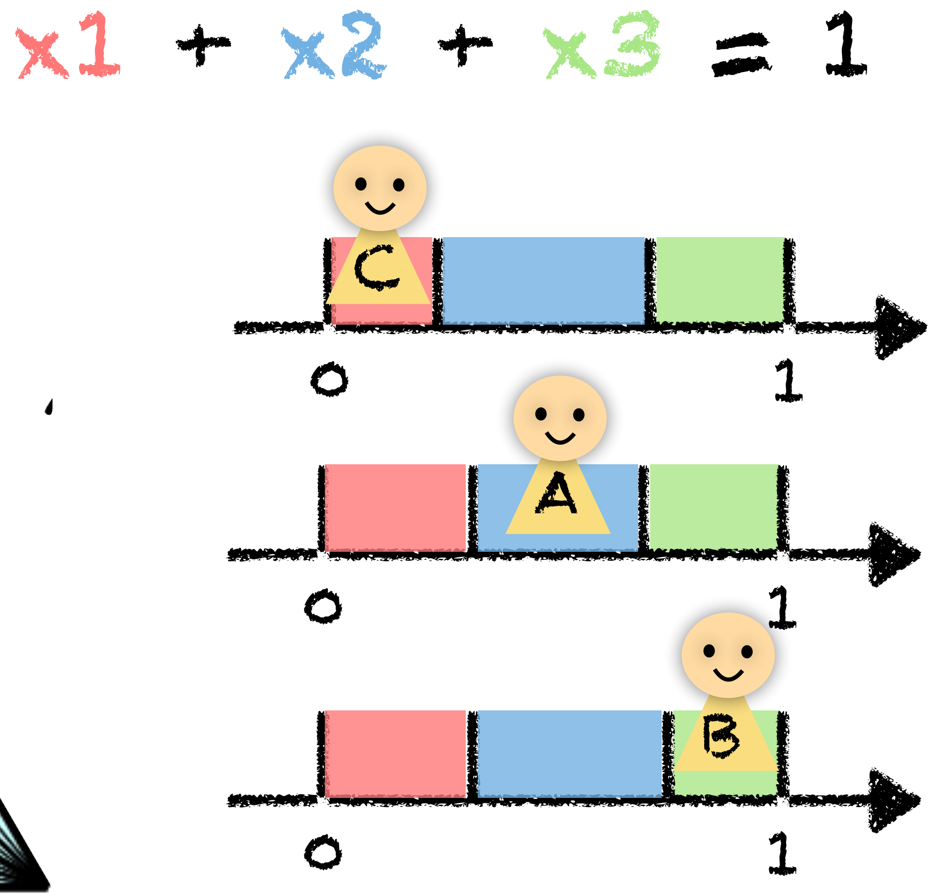


Sperner's Lemma and cake-cutting

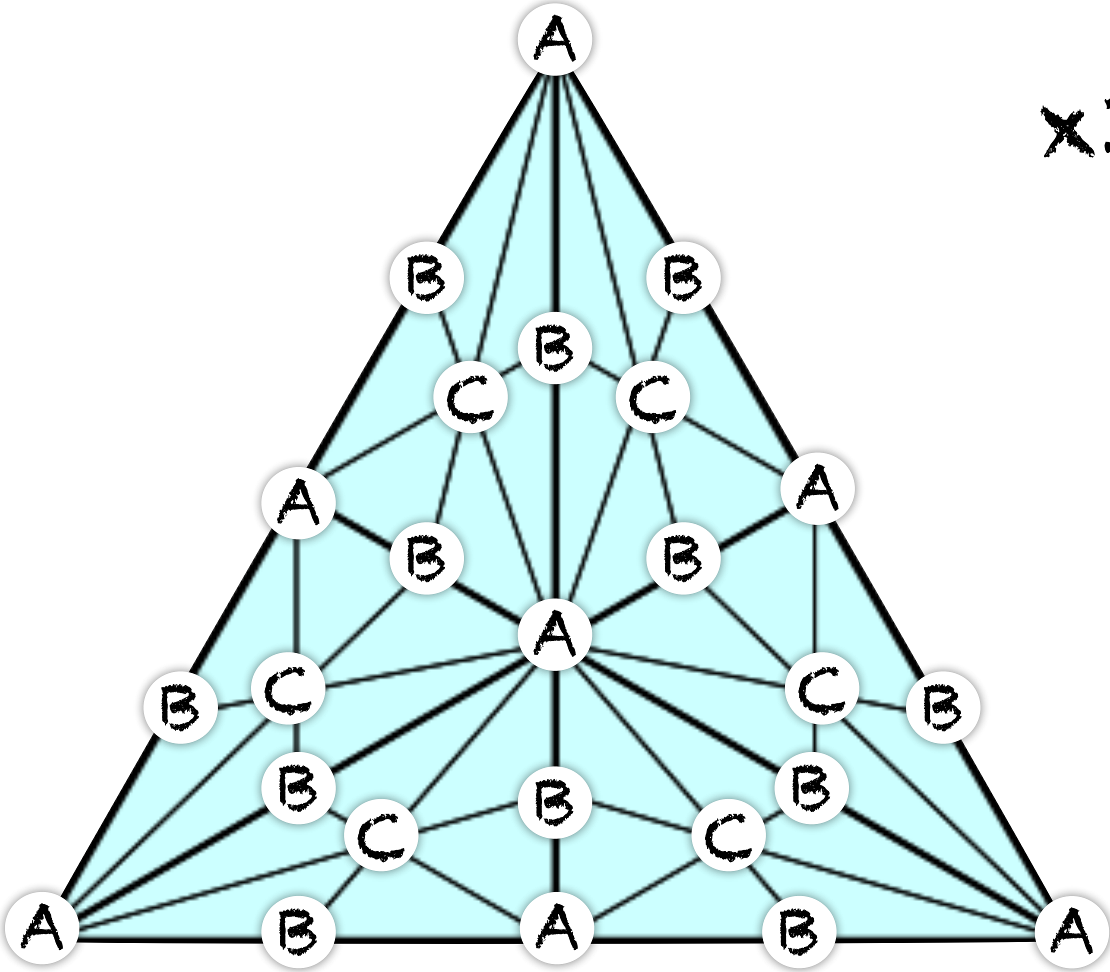
4. Make the triangulation finer.





Can we use Sperner?

- One may apply Sperner-type argument.



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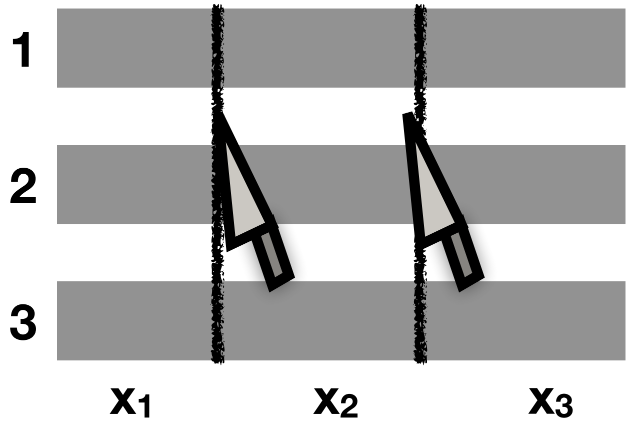

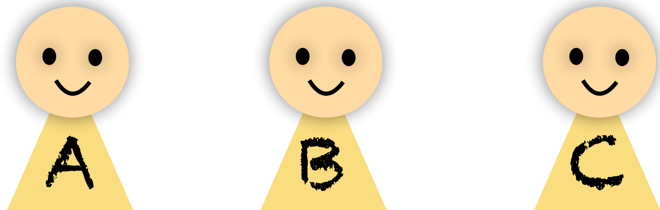

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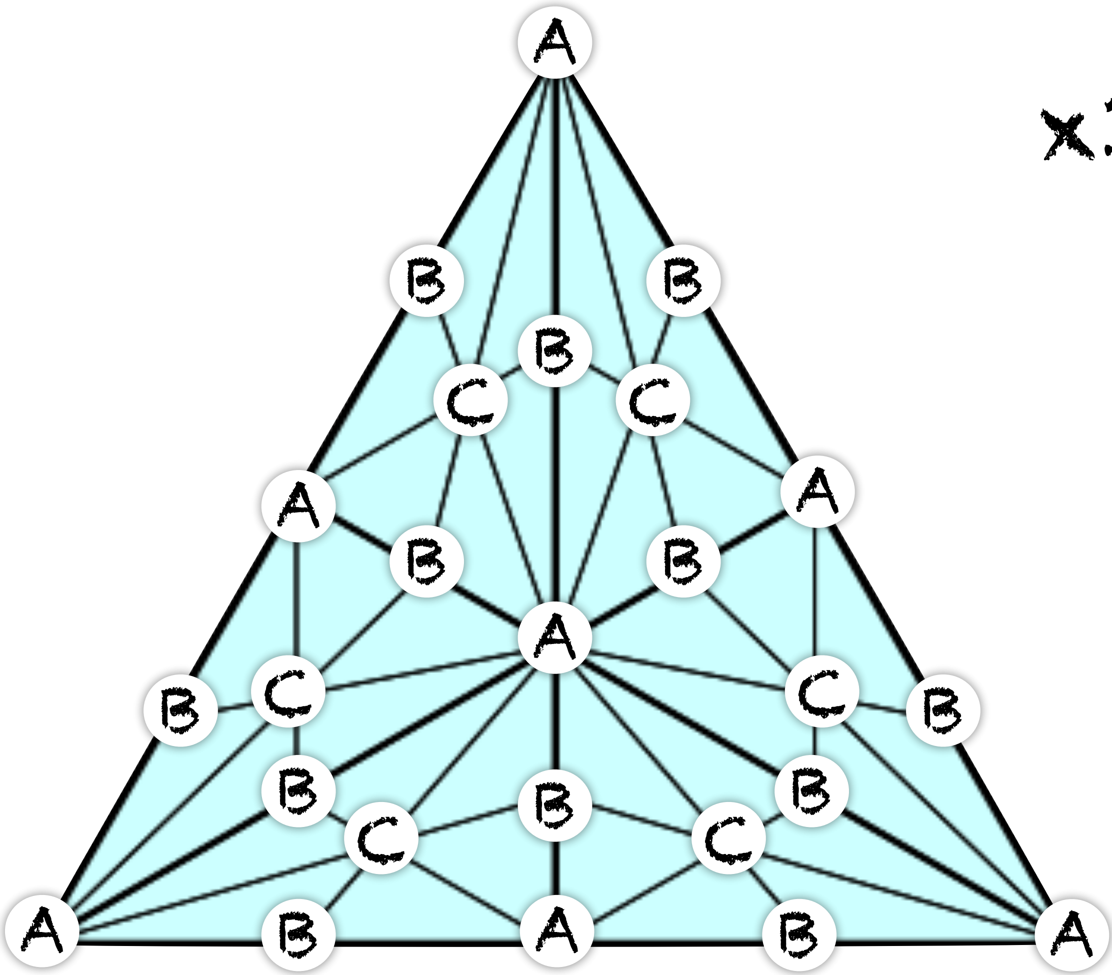

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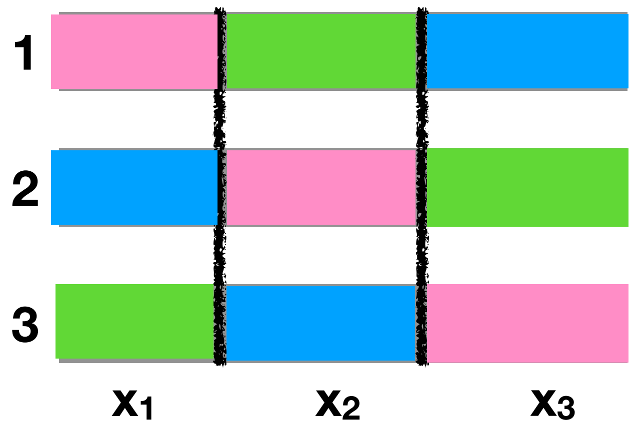

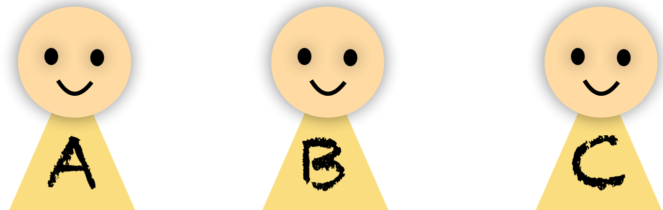

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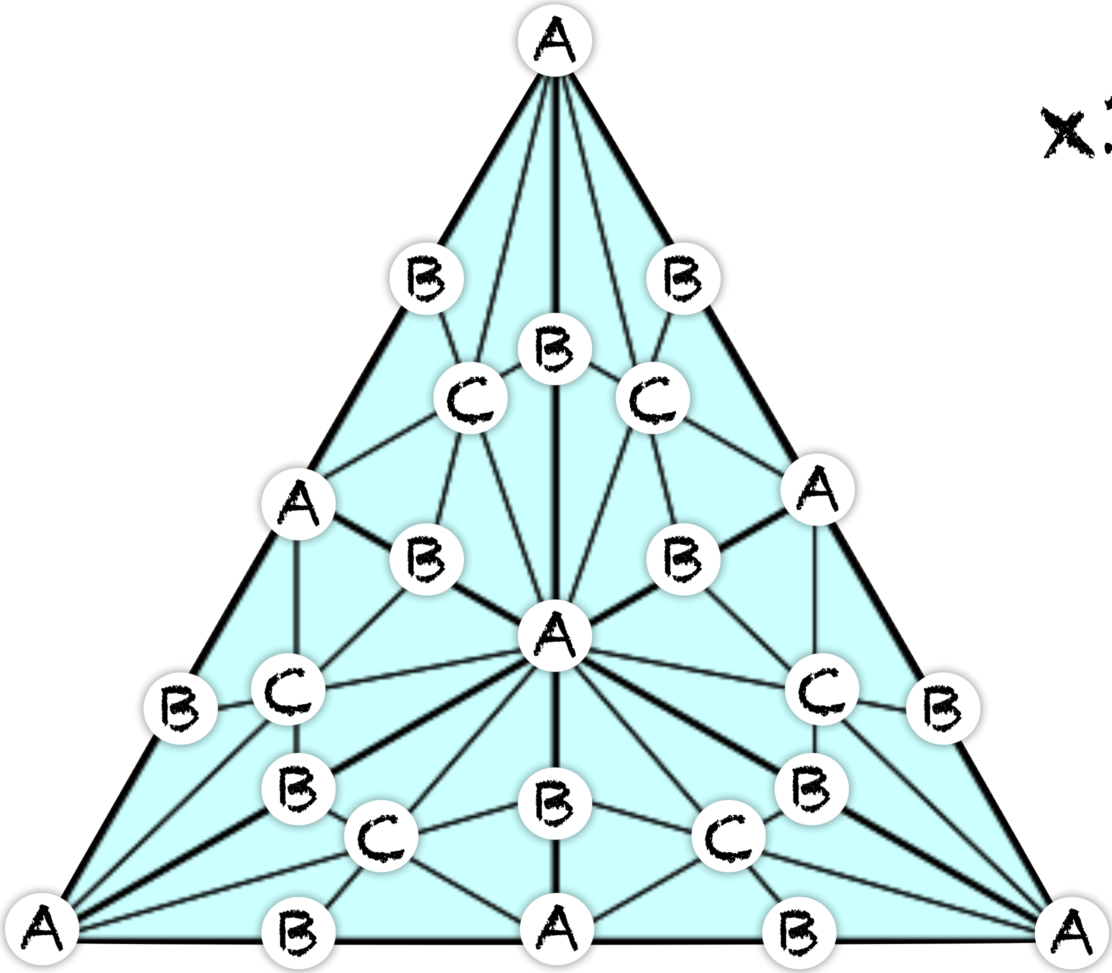
  → The boundary condition may not be satisfied.



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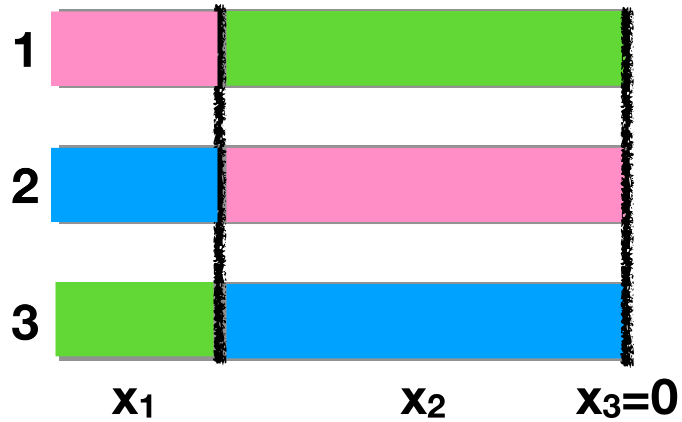
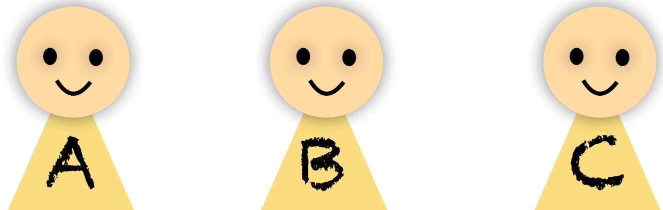
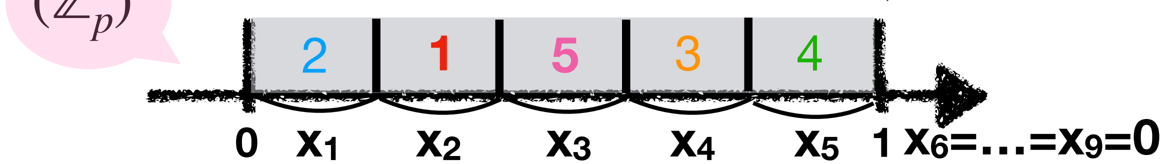
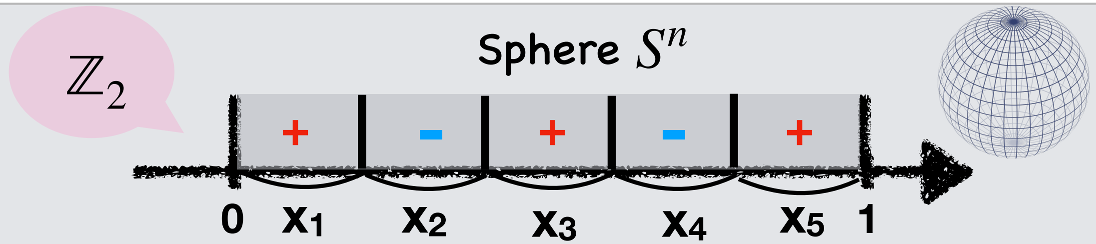
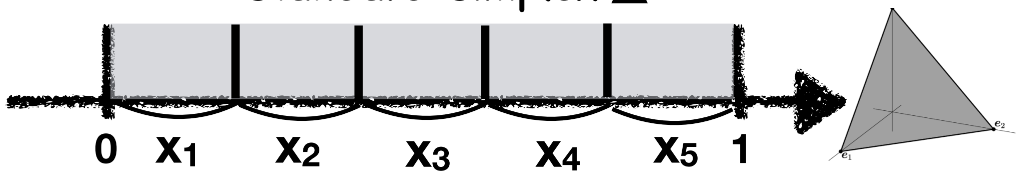




Can we use Sperner?

- Instead of Sperner, we use a more general Borsuk-Ulam-type theorem proven by Volovikov (1996)

Applications	Fixed Point Thm	Configuration Space
EF division of a partially poisoned cake [Jojić, Panina, Živaljević, 2021]	Volovikov Theorem	Chessboard Complex $\Delta_{2n-1,n}$ 
Consensus-Halving [Simmons-Su, 2003]	Tucker's lemma (Borsuk-Ulam)	Sphere S^n 
EF division of a tasty cake [Stromquist, 1980, Woodall 1980, Su 1999]	Sperner's lemma (Brouwer)	Standard Simplex Δ^{n-1} 

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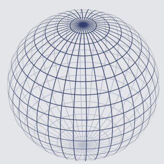
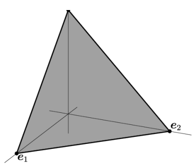
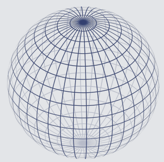
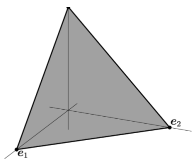
Applications	Fixed Point Thm	Configuration Space
EF division of a partially poisoned cake [Jojić, Panina, Živaljević, 2021]	Volovikov Theorem	$G = ((\mathbb{Z}_p)^k, +)$ Chessboard Complex $\Delta_{2n-1,n} = \{ (g_1x_1, g_2x_2, \ldots, g_{2n-1}x_{2n-1}) \mid x \in \mathbb{R}^n_+, \sum_i x_i = 1,$ $g_i \in G \cup \{\varnothing\}, x_i = 0$ **if** $g_i = \varnothing$, n **of** g_i **are non-empty distinct labels** $\}$
Consensus-Halving [Simmons-Su, 2003]	Tucker's lemma (Borsuk-Ulam)	Sphere $S^n = \{ x \in \mathbb{R}^n \mid \sum_i \lvert x_i \rvert = 1 \}$ 
EF division of a tasty cake [Stromquist, 1980, Woodall 1980, Su 1999]	Sperner's lemma (Brouwer)	Standard Simplex $\Delta^{n-1} = \{ x \in \mathbb{R}^n_+ \mid \sum_i x_i = 1 \}$ 

Can we use Sperner?

- Instead of Sperner, we use a more general Borsuk-Ulam-type theorem proven by Volovikov (1996)

Applications	Fixed Point Thm	Configuration Space
EF division of a partially poisoned cake [Jojić, Panina, Živaljević, 2021]	Volovikov Theorem	Chessboard Complex $\Delta_{2n-1,n} = \{ (g_1x_1, g_2x_2, \ldots, g_{2n-1}x_{2n-1}) \mid x \in \mathbb{R}^n_+, \sum_i x_i = 1,$ $g_i \in G \cup \{\varnothing\}, x_i = 0$ **if** $g_i = \varnothing,$ ... **of** ... **are non-empty distinct labels** $\}$
Consensus-Halving [Simmons-Su, 2003]	Tucker's lemma (Borsuk-Ulam)	$S^n = \{ x \in \mathbb{R}^n \mid \sum_i \lvert x_i \rvert = 1 \}$ 
EF division of a tasty cake [Stromquist, 1980, Woodall 1980, Su 1999]	Sperner's lemma (Brouwer)	Standard Simplex $\Delta^{n-1} = \{ x \in \mathbb{R}^n_+ \mid \sum_i x_i = 1 \}$ 

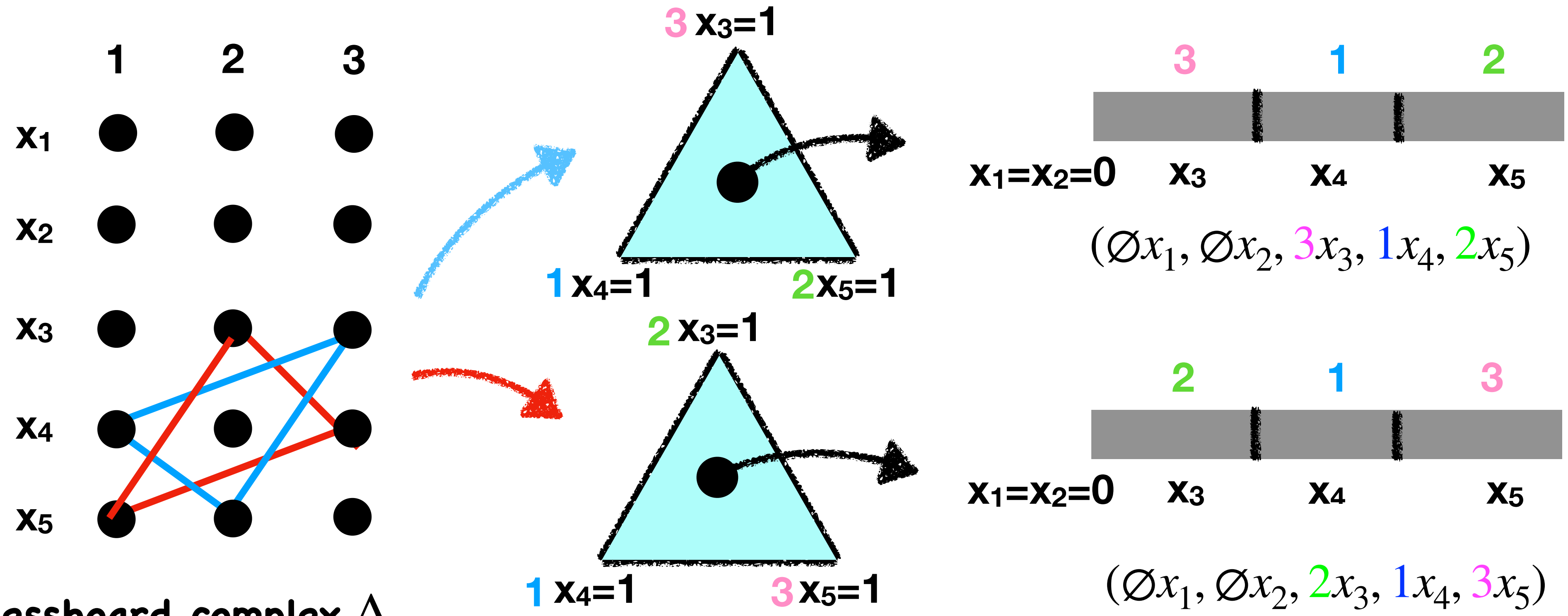
$(2x_1, 1x_2, 5x_3, \varnothing x_4, 4x_5, 3x_6, \varnothing x_7 \ldots, \varnothing x_9)$

$(+x_1, -x_2, +x_3, -x_4, +x_5)$

$(x_1, x_2, x_3, x_4, x_5)$

Proof Idea for m=3 and n=3

- Our configuration space encodes not only diagonal shares using $n-1$ long knives but also possible permutations of indices.



Chessboard complex $\Delta_{2n-1,n}$

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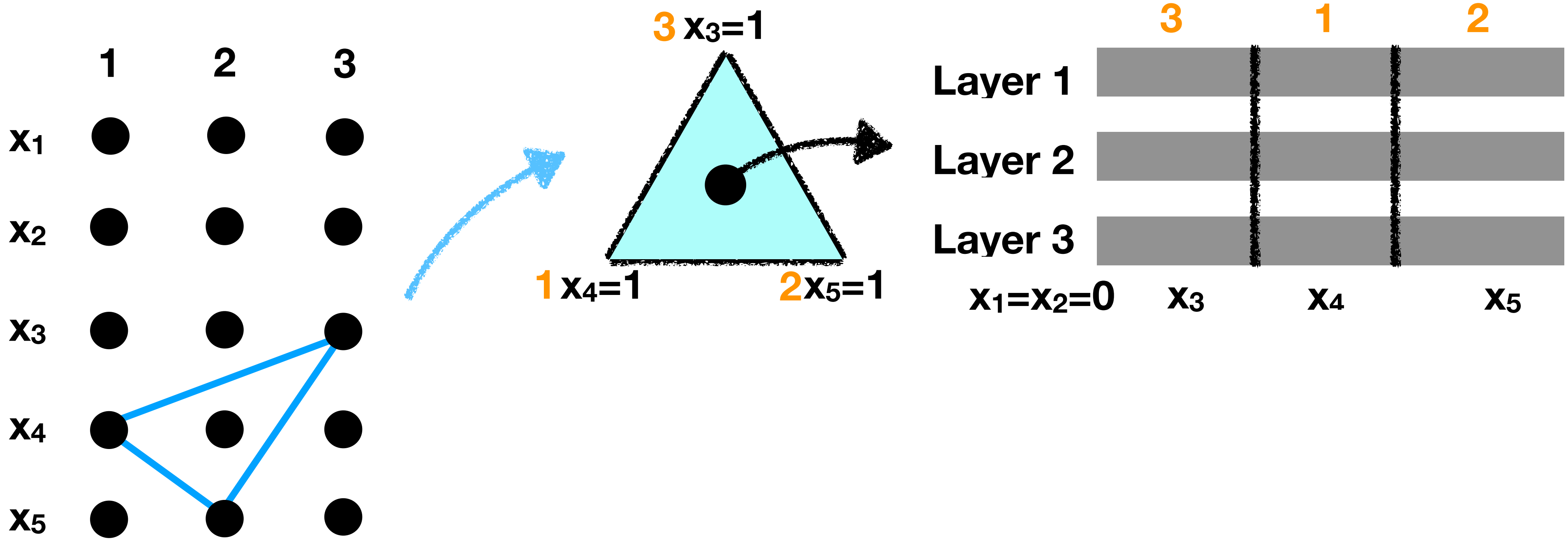

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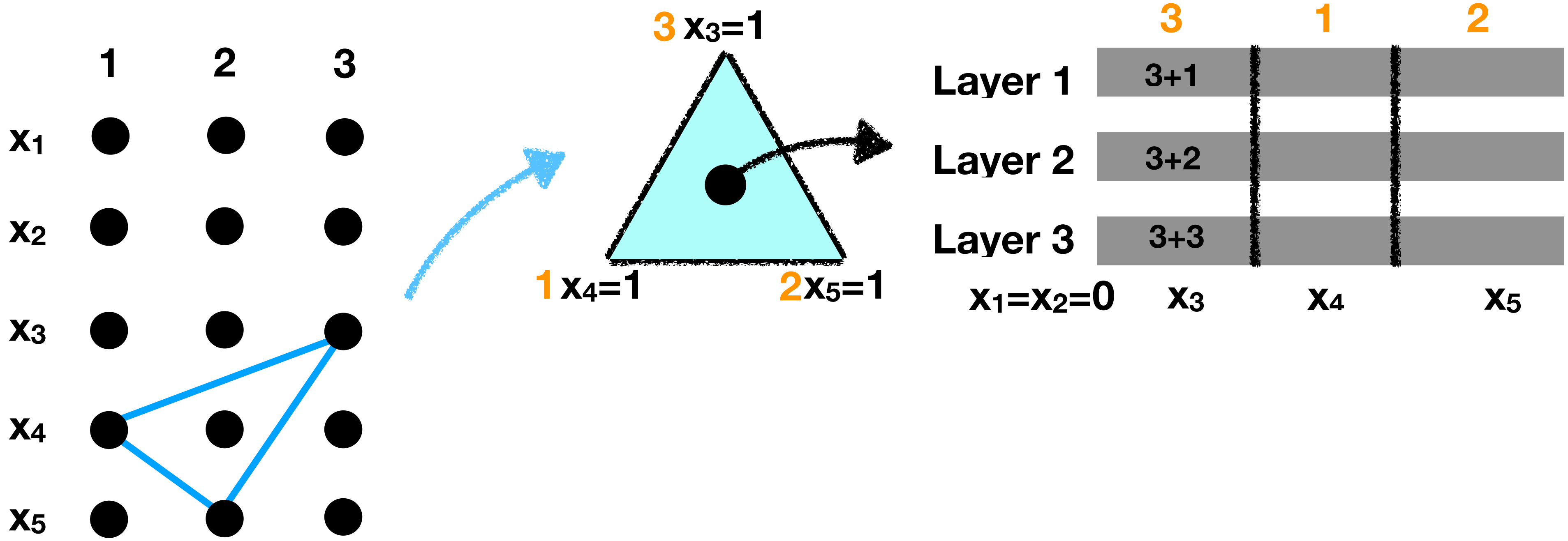

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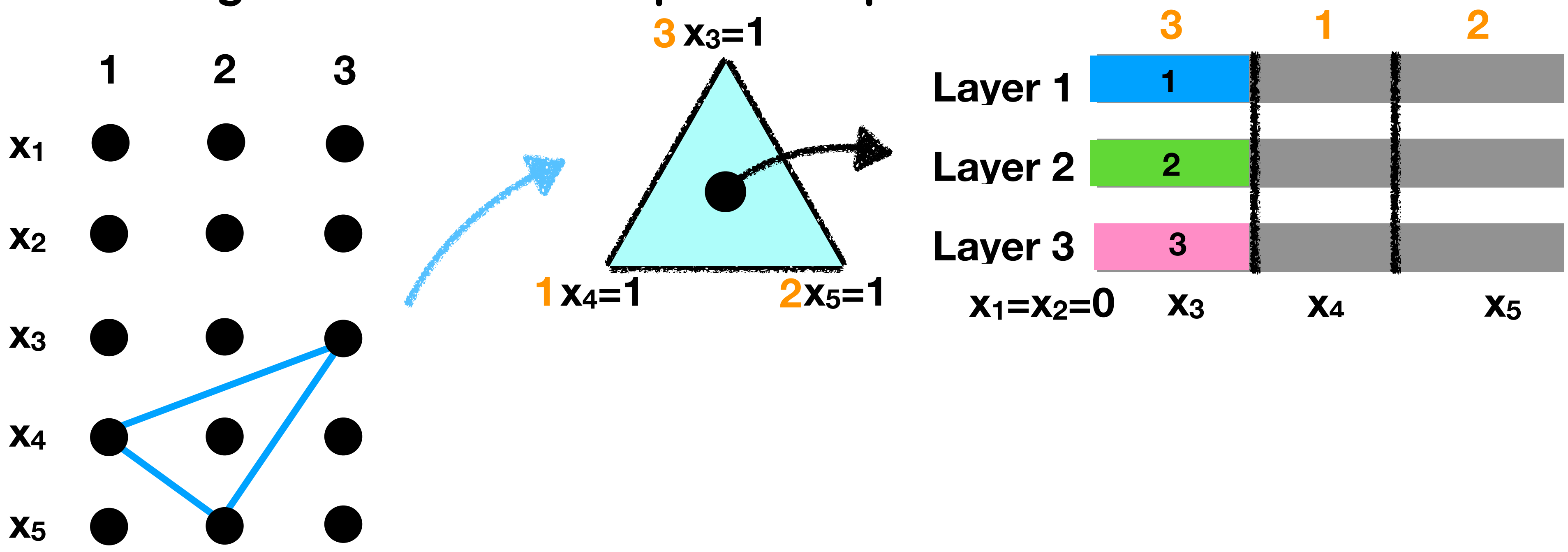

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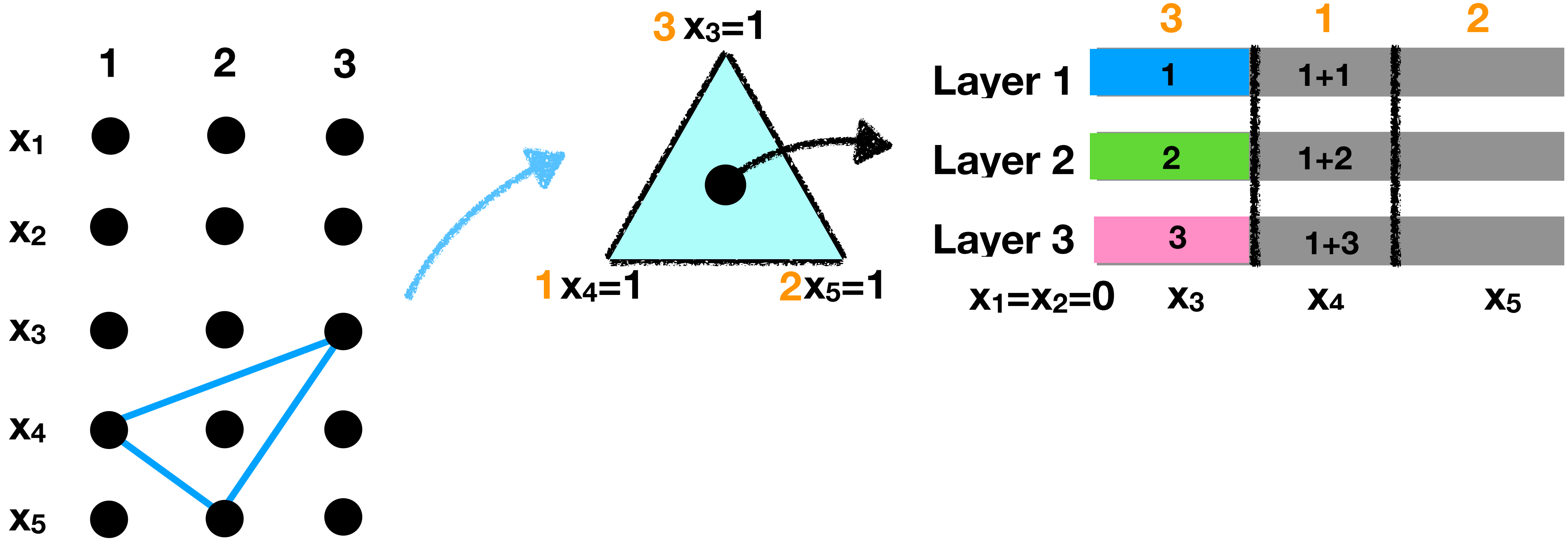

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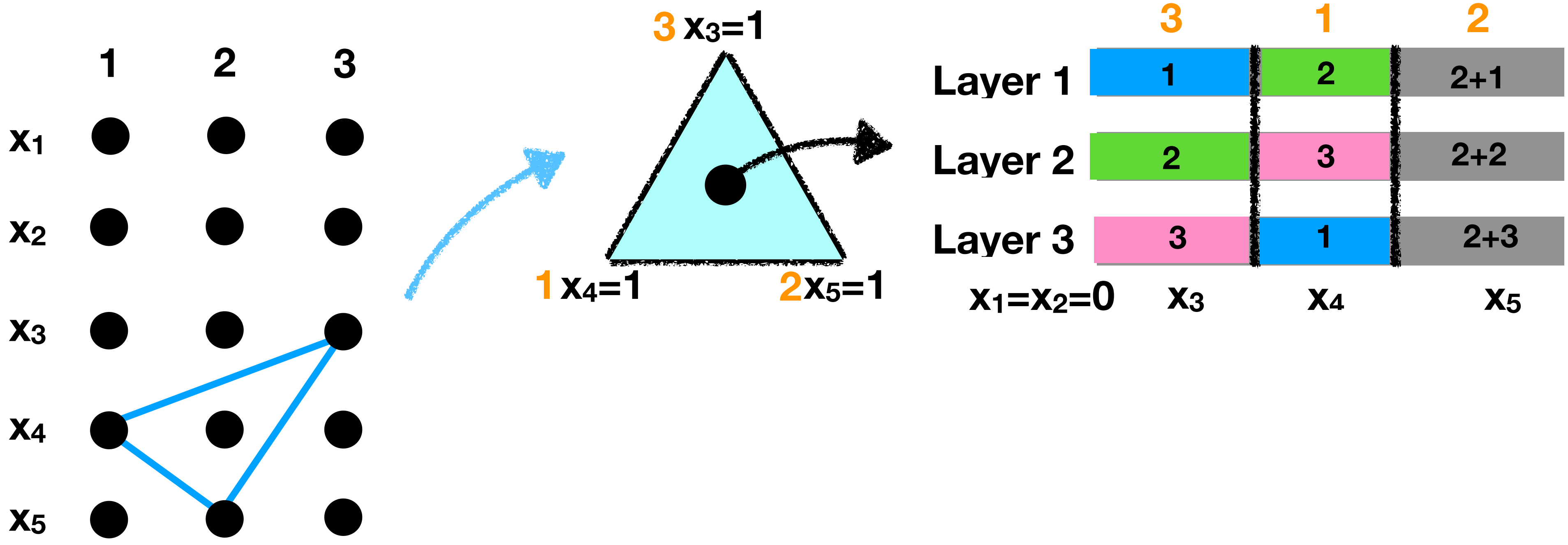

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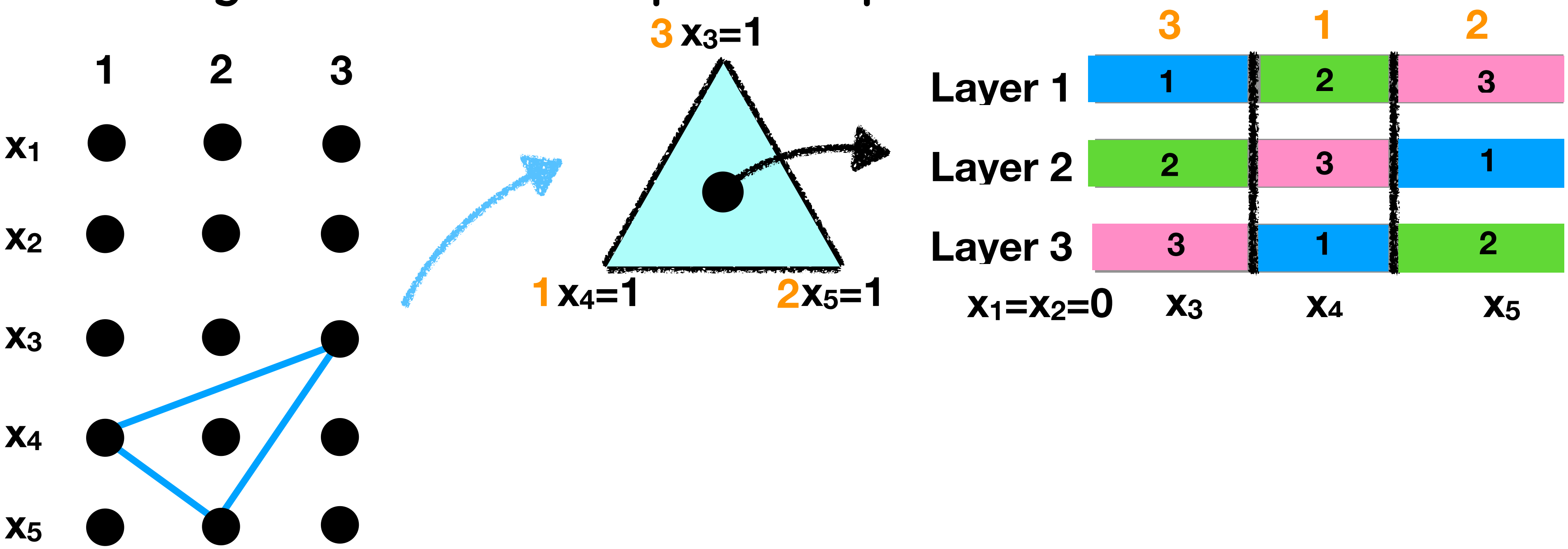

Chessboard complex $\Delta_{2n-1,n}$

Proof Idea for m=3 and n=3

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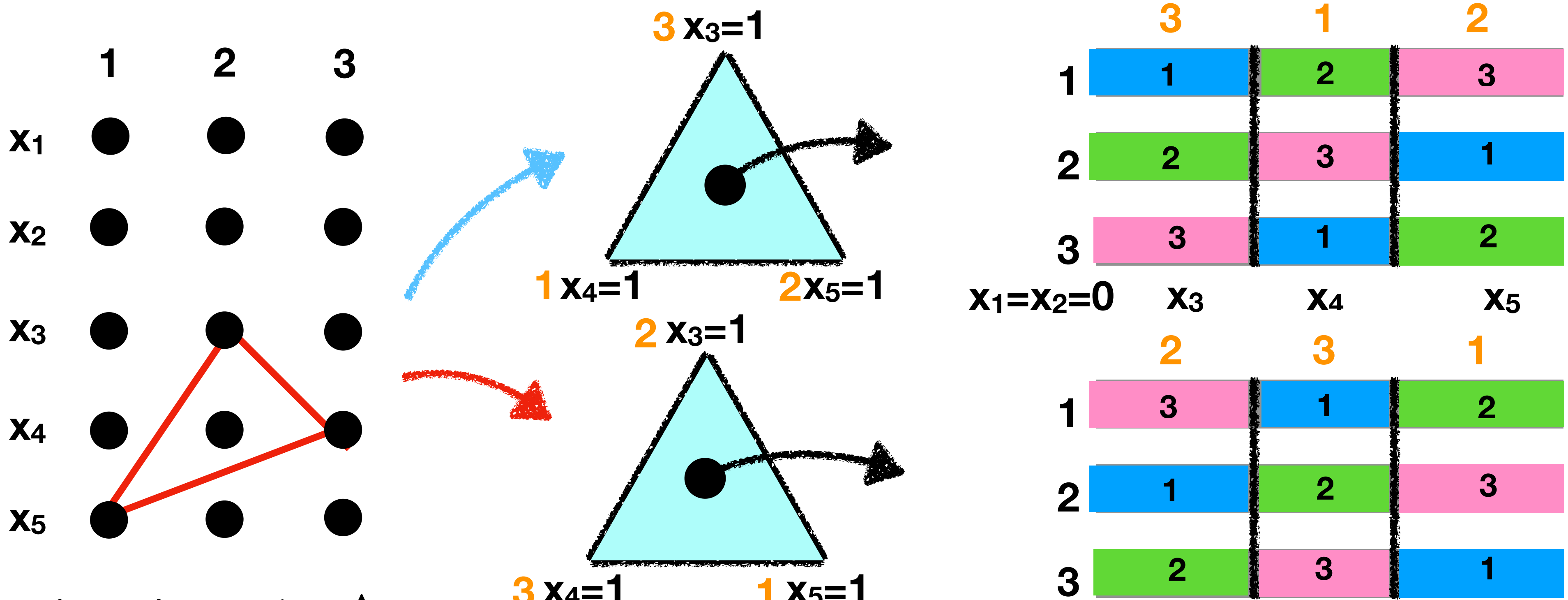
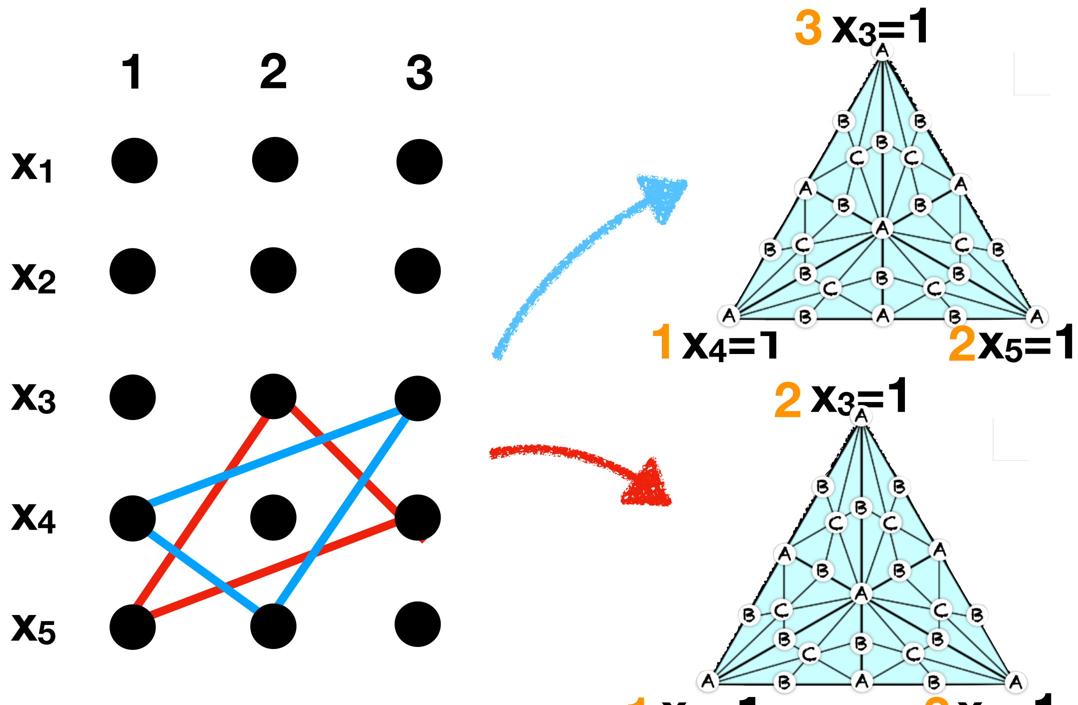


Chessboard complex $\Delta_{2n-1,n}$

Proof Idea for m=3 and n=3

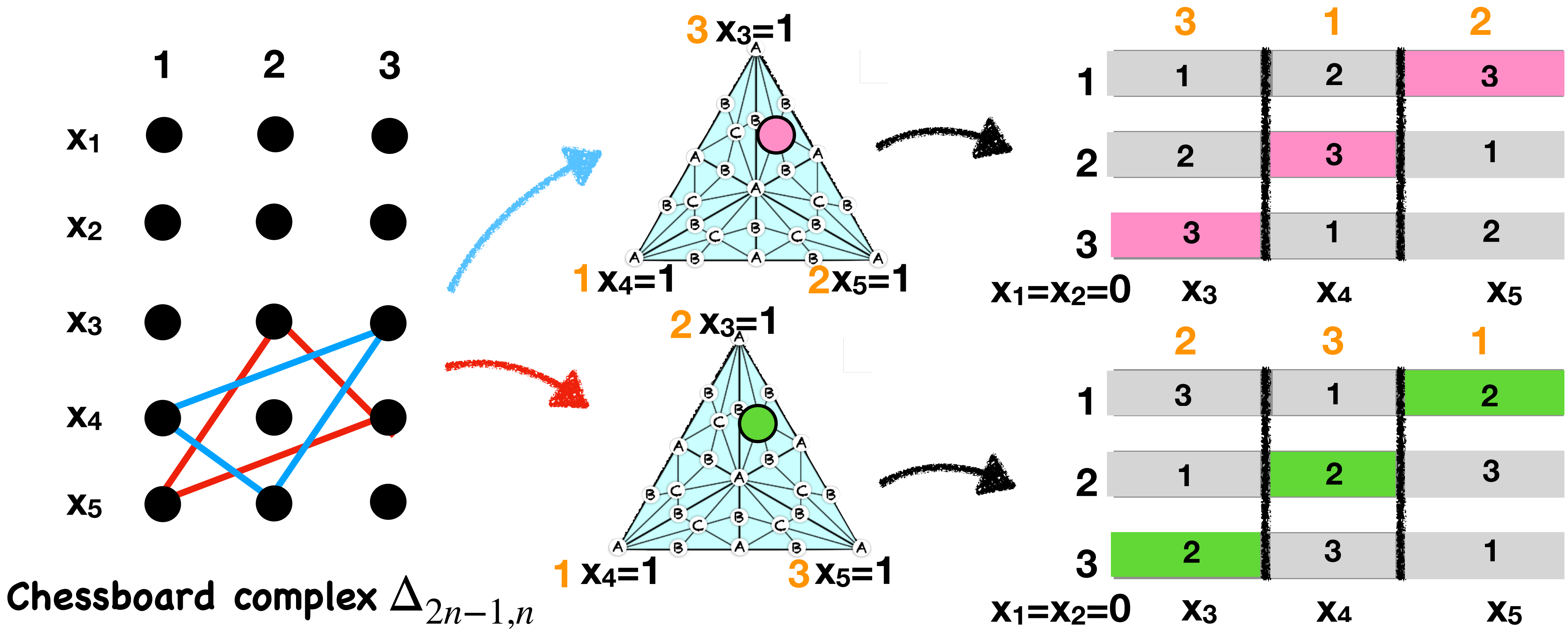
1. Triangulate the simplex and assign owner label so that every small triangle receives different agent labels.



Chessboard complex $\Delta_{2n-1,n}$

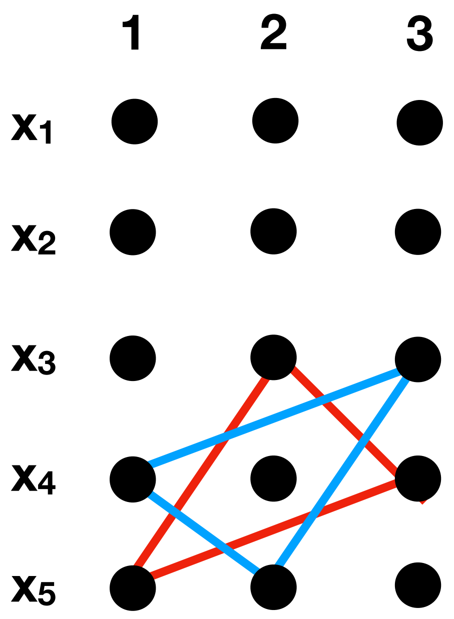
Proof Idea for m=3 and n=3

2. Each owner assigns the color of the favorite piece.

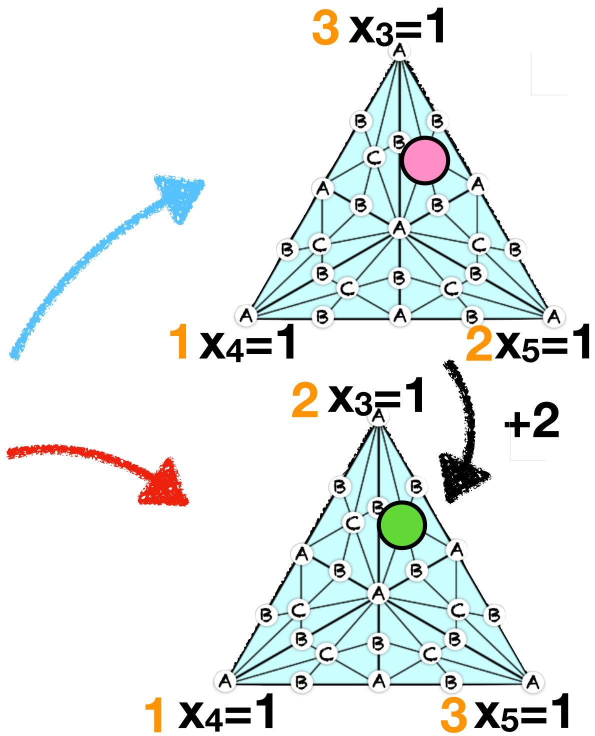


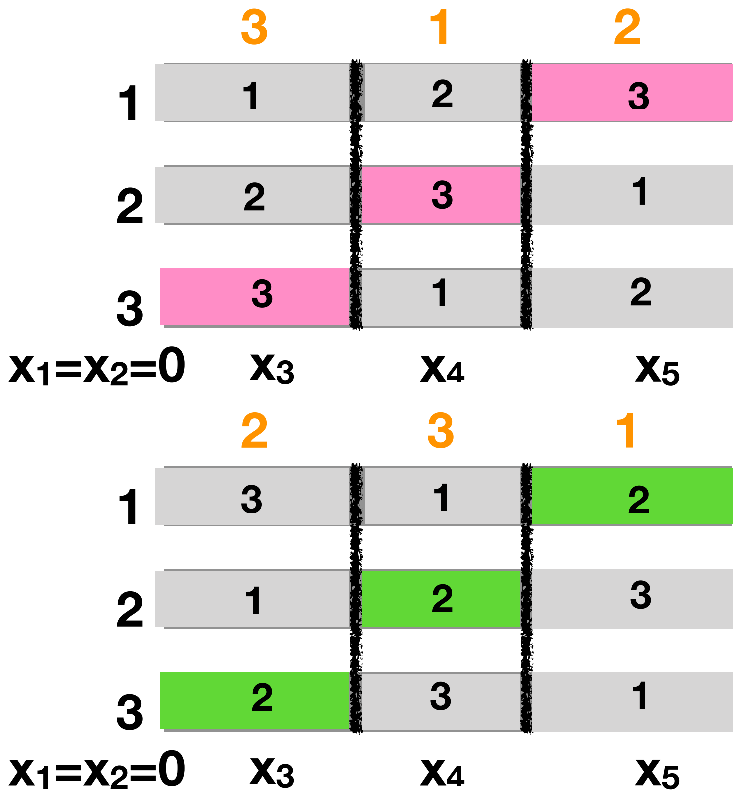
Proof Idea for m=3 and n=3

3. Apply Volovikov's theorem and get a fully colored triangle.



Chessboard complex $\Delta_{2n-1,n}$





- Volovikov's theorem ensures that for any G-equivariant coloring of G-invariant triangulation of $\Delta_{2n-1,n}$ with elements of $G=((Z_p)^k,+)$, there is a fully colored simplex (p is a prime number).

Discussion

- The case when the number of agents is not a prime power.
  - Limitation of the approach based on equivariant topology. (Volovikov's theorem doesn't hold.)
  - Avvakumov and Krasev (2020): EF division exists when agents have identical valuations (not necessarily monotone).
- Computational complexity.
  - No constructive proof of Volovikov's theorem. The precise complexity class is open.