Axiomatization of Meaningful Solutions:
Connecting Cooperative Game Theory to Social Network Analysis

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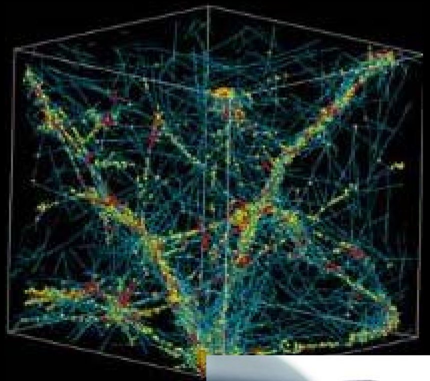
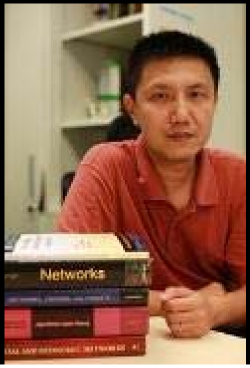




Joint work with Wei Chen (MSR Asia)
Hanrui Zhang (Duke/CMU)

Game Theory

Two Basic Paradigms

- **Non-Cooperative Games:**
  - competition between individual players

  Solution Concepts: Nash equilibrium

  Applications: Market exchange economics

- **Cooperative Games:**
  - group of players
  - coalitional games

  external enforcement of cooperative behavior

  Applications: political science, formation of companies,
  payoffs of coalitions

Data in Cooperative Game Theory

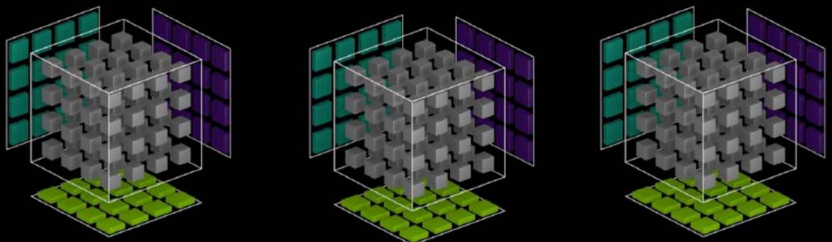
- **Grand Coalition** - Player Set: [*N*]
- **Group Utilities:**

$$\sigma\colon\ 2^N \rightarrow \mathbb{R}\,, \sigma(\emptyset) = 0$$

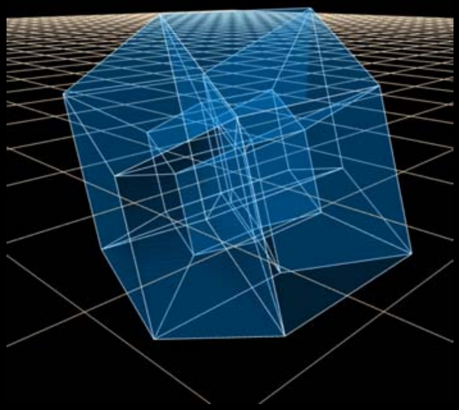
allocate the payoff among the players in some fair way

Mathematical Spaces of Data Models

Three-Player Four-Strategy Games



Five-Player Cooperative Games



Data in Cooperative Game Theory

- Grand Coalition - Player Set: [*N*]
- Group Utilities:

$$\sigma\colon\ 2^N \rightarrow \mathbb{R}\,, \sigma(\emptyset) = 0$$

Features:

- Big Model
- Weighted Hypergraphs

allocate the payoff among the players in some fair way

A Basic Solution Concept

- **for measuring Individual contribution to coalition games**
- **for fair allocation of global values**

when given group utilities:

$$\sigma:\ 2^{N} \rightarrow \mathbb{R}\,,\sigma(\emptyset) = 0$$

allocate the payoff among the players in some fair way

Dimensionality Reduction

- **Data - group utilities:**

$$\sigma\colon\ 2^N \to \mathbb{R}\,, \sigma(\emptyset) = 0,$$

$$\text{i.e.}, \mathbb{R}^{2^N}$$

- **Solution Concept:**

$$\phi\colon \mathbb{R}^{2^N} \to \mathbb{R}$$

allocate the payoff among the players in some fair way

Dimensionality Reduction

- **Data - group utilities:**

$$\sigma\colon 2^N \to \mathbb{R}, \sigma(\emptyset) = 0,$$

$$\text{i.e.}, \mathbb{R}^{2^N}$$

- **Solution Concept:**

$$\phi\colon \mathbb{R}^{2^N} \to \mathbb{R}$$

$$\phi \in \mathbb{R}^N$$

Shapley Values

$$\mathbf{SV}_{\sigma}(k) = E_{\pi}[\, \sigma(S_{\pi,k} + k) - \sigma(S_{\pi,k}) \,]$$

$S_{\pi,k}$: *players placed before k according to* π

Expected Marginal Contribution

Shapley Values

Why is this meaningful?

Axiomatic Properties

- **Efficiency**

$$\sum_{k} \mathrm{SV}_{\sigma}(k) = \sigma([N])$$

- **Symmetry**

if $\forall S$, $\sigma(S + i) = \sigma(S+j)$, then $\mathrm{SV}_{\sigma}(i) = \mathrm{SV}_{\sigma}(j)$

- **Linearity**

for an group values σ and τ, $\mathrm{SV}_{\sigma+\tau} = \mathrm{SV}_{\sigma} + \mathrm{SV}_{\tau}$

- **Null Player**

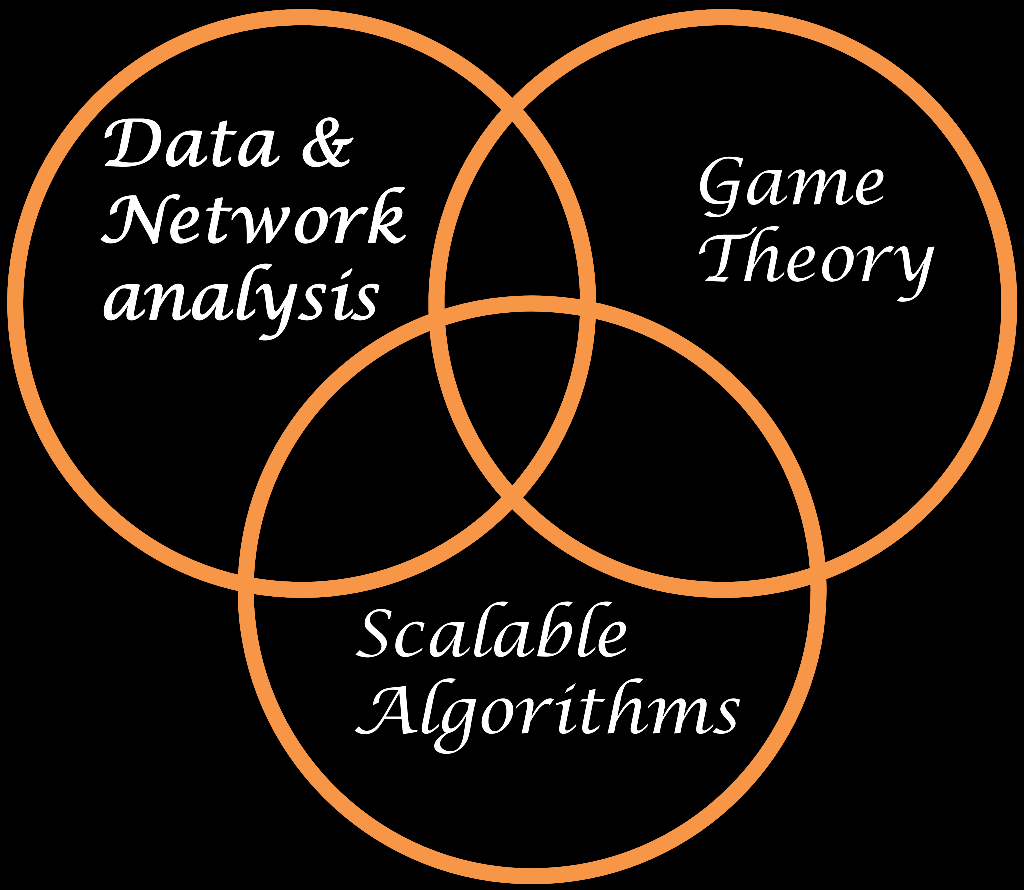
if $\forall S$, $\sigma(S + k) = \sigma(S)$, then $\mathrm{SV}_{\sigma}(k) = 0$

Shapley's Axiomatic Characterization

$$\phi: \mathbb{R}^{2^N} \to \mathbb{R}$$

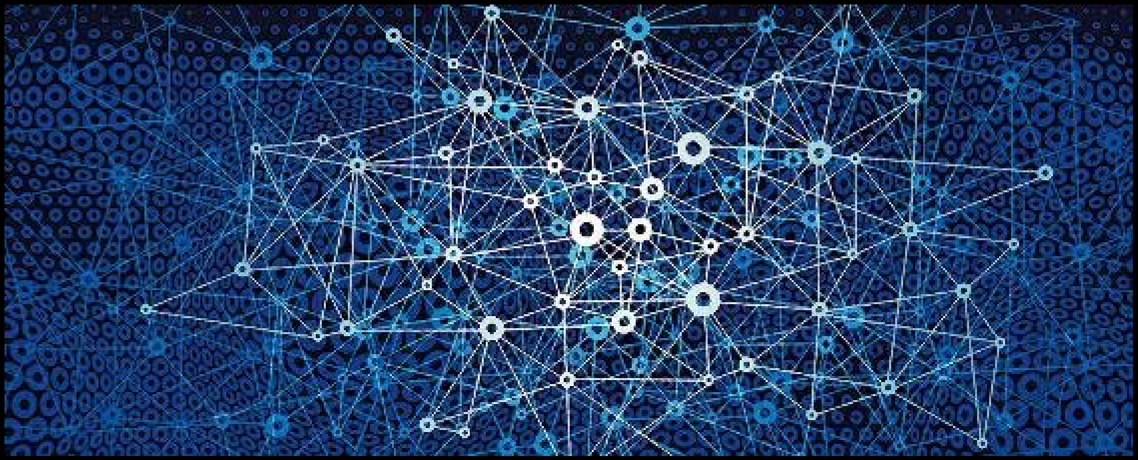
- **Efficiency**
- **Symmetry**
- **Linearity**
- **Null Player**

Applications to Data & Network Analysis



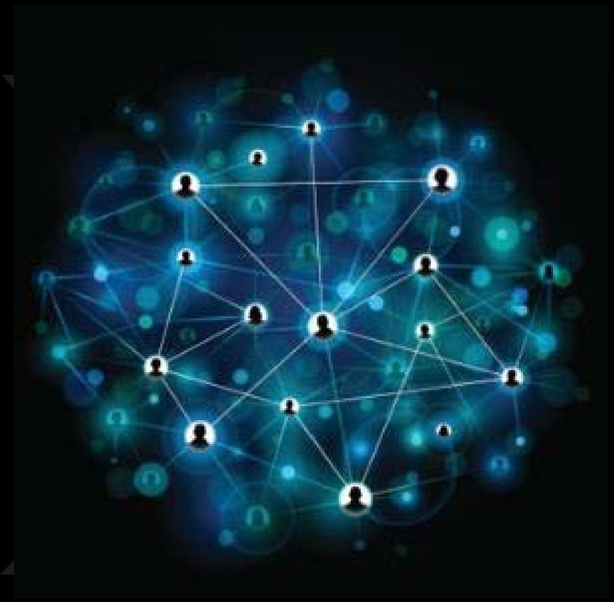
Graph Theory in the Age of Networks

- Graph Model
  - Nodes: Webpages, Internet routers, or people
  - Edges: links, connections, or friends



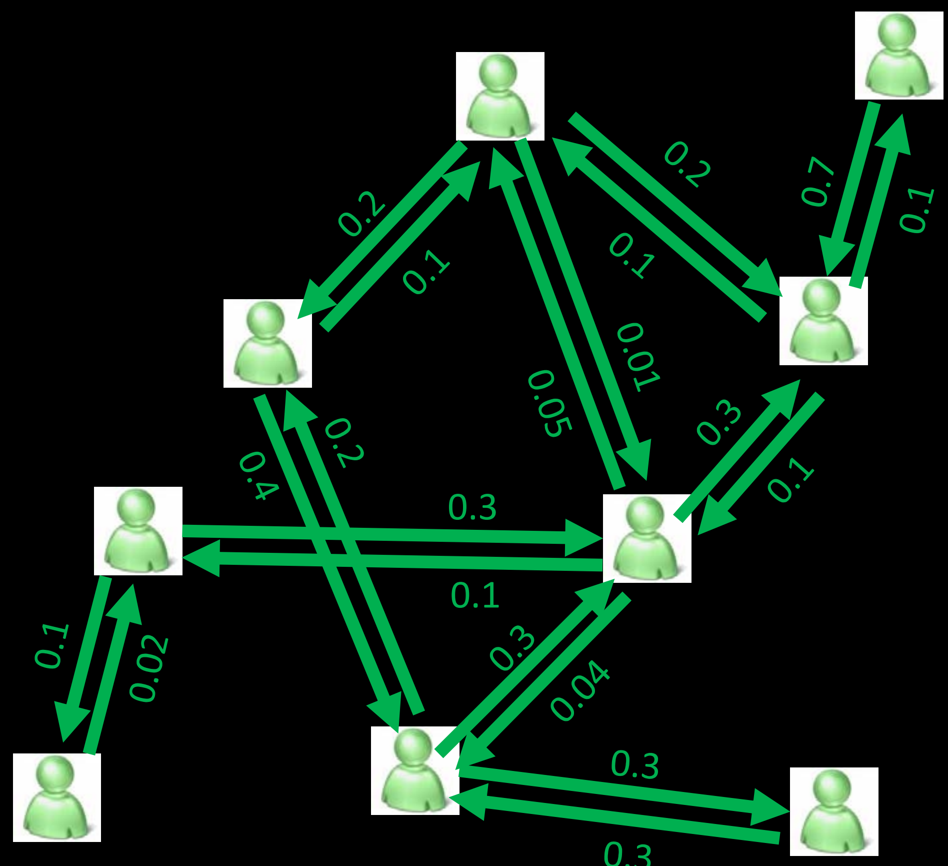
Networks are more than their graph representations

Network Data: Rich and Multi-Faceted



Independent Cascade Model

Kempe-Kleinberg-Tardos

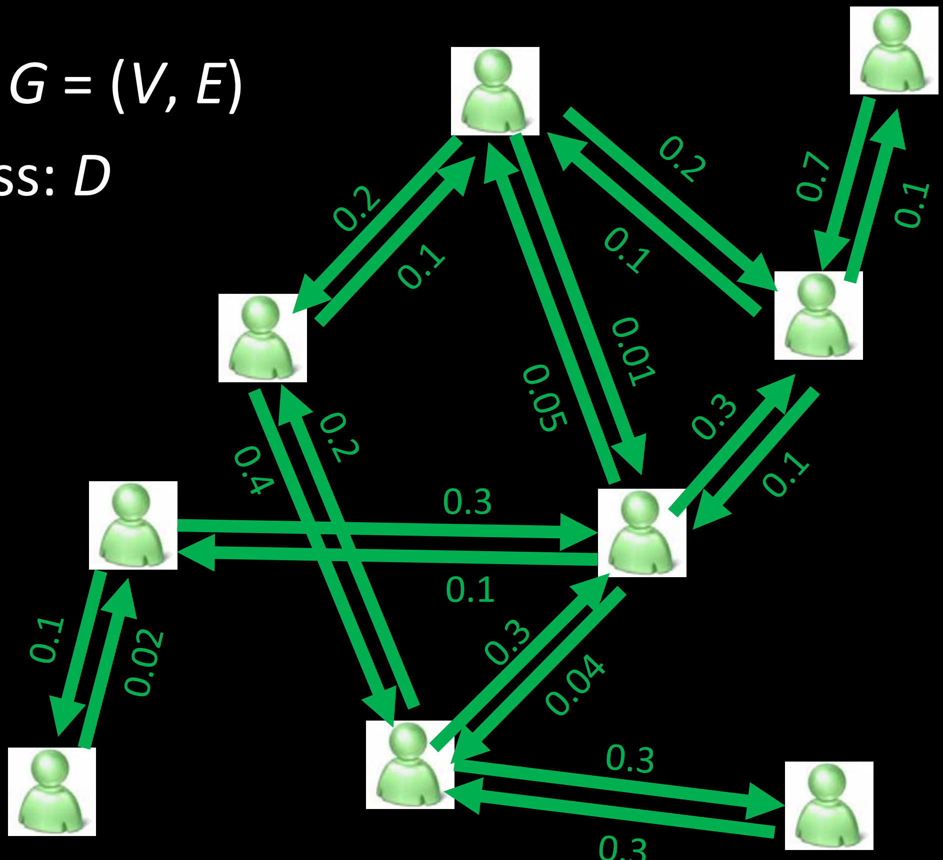


Independent Cascade Model

Kempe-Kleinberg-Tardos

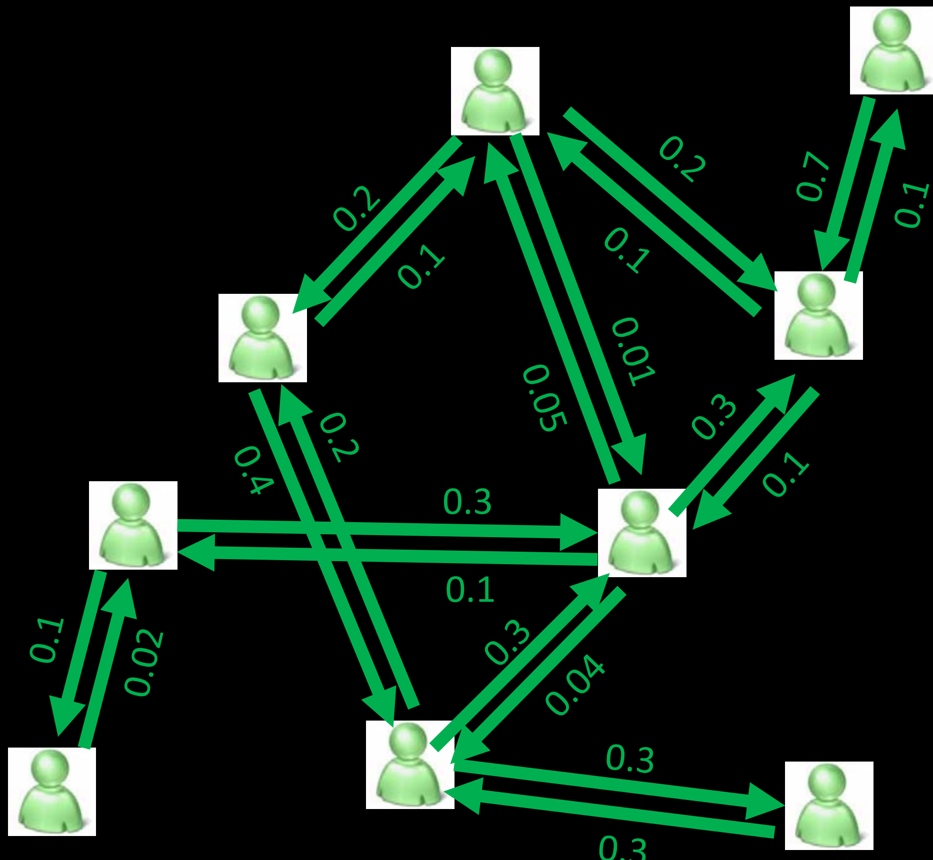
Social Network: $G = (V, E)$

Influence Process: D



Viral Marketing

Domingos and Richardson



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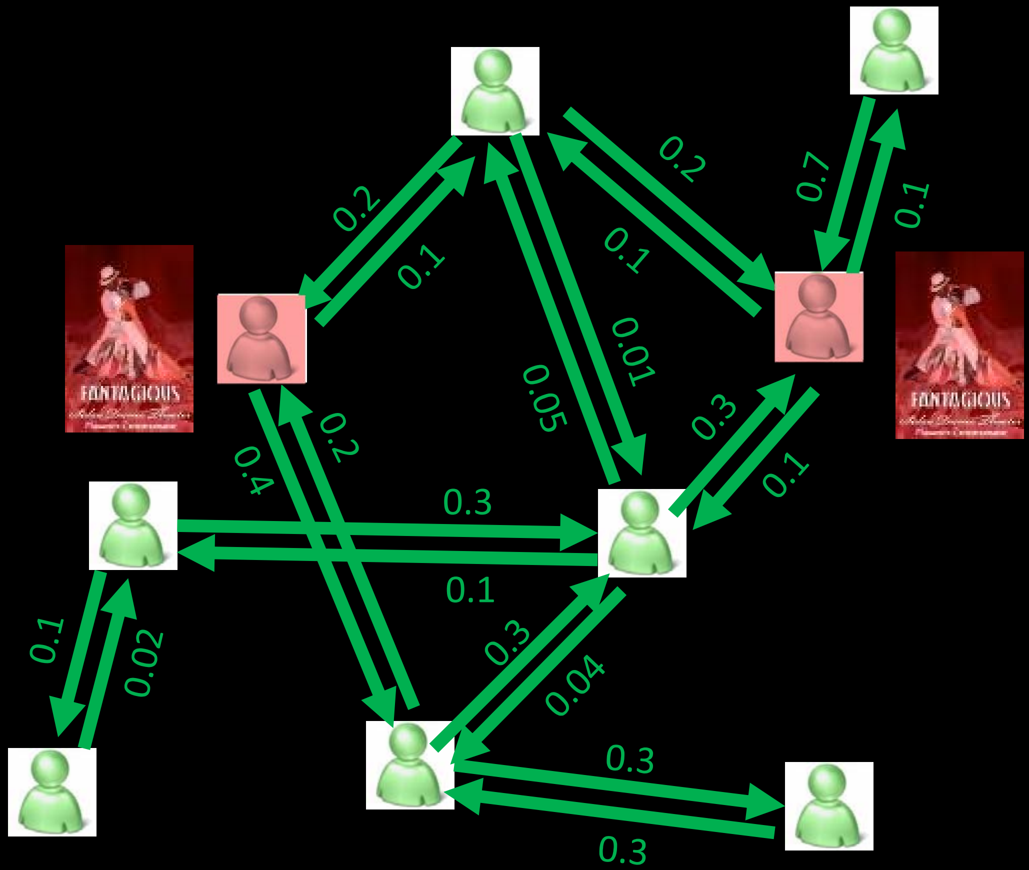

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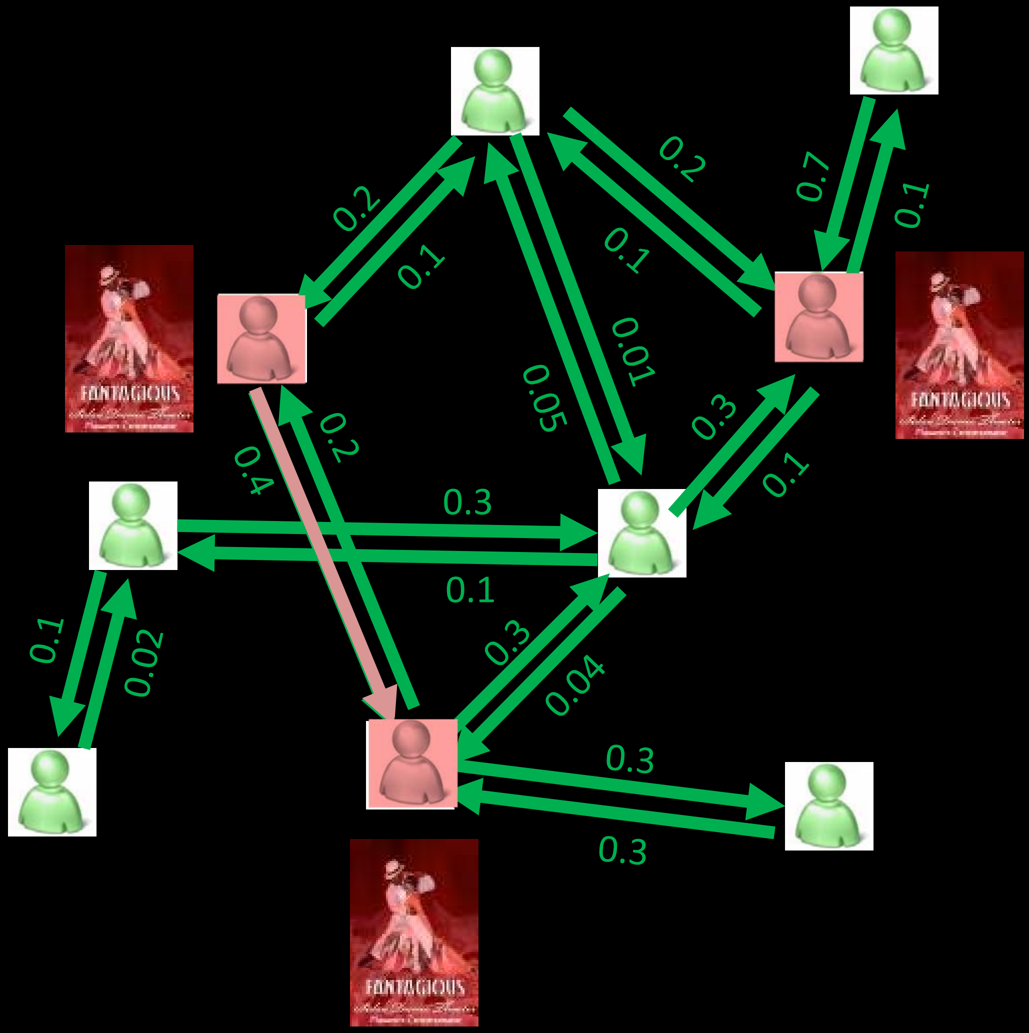

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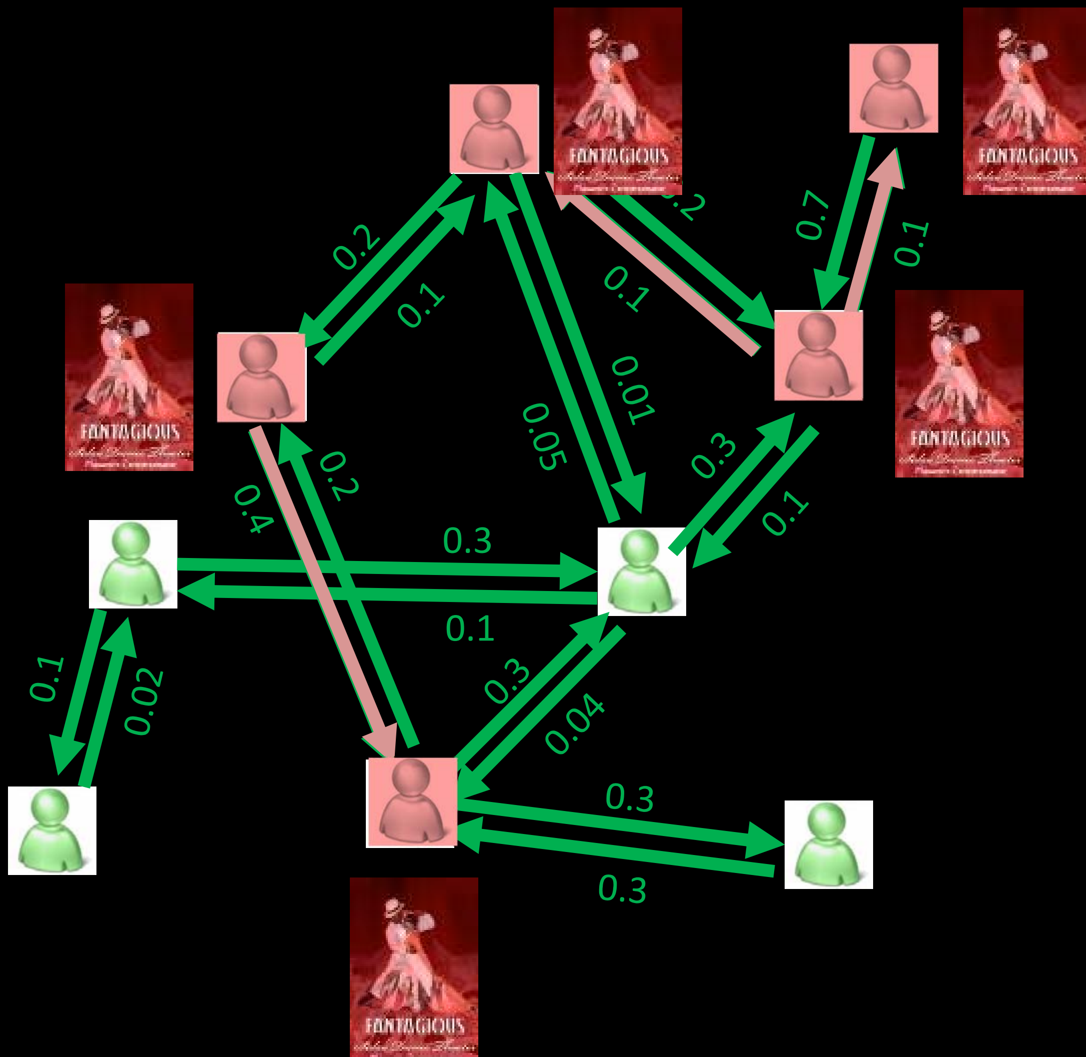

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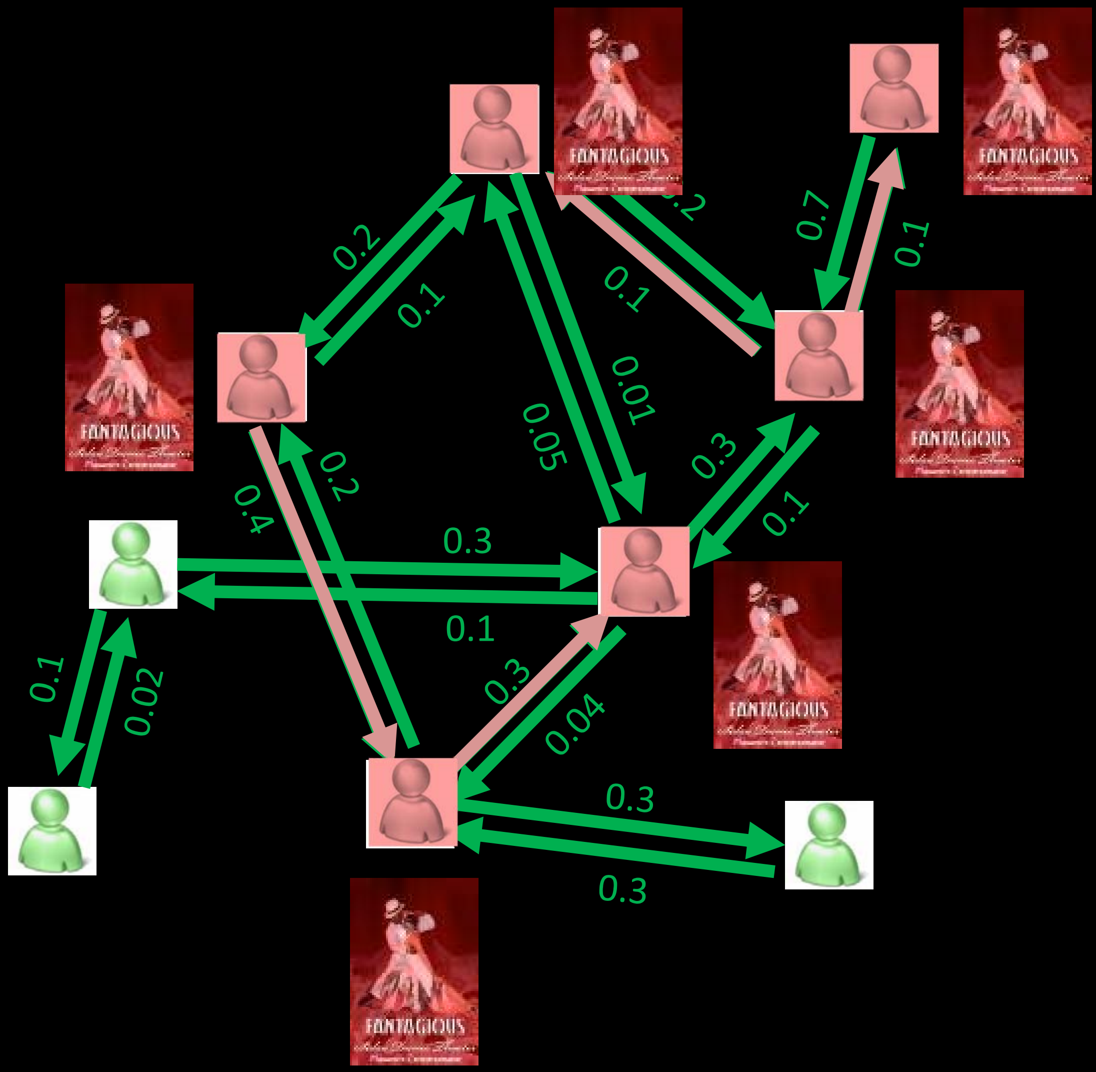


Social Influence

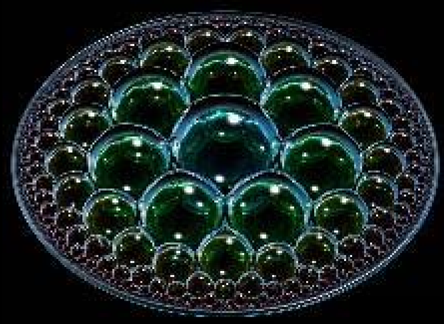
Understanding Multi-Faceted Network Data

What is the impact of an influence process on network centrality?

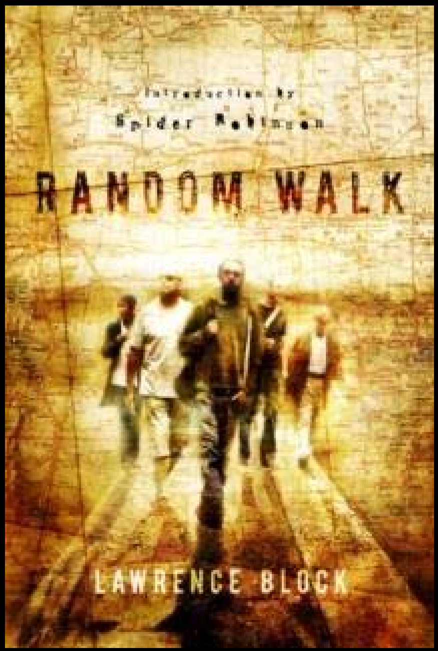
Network Centrality

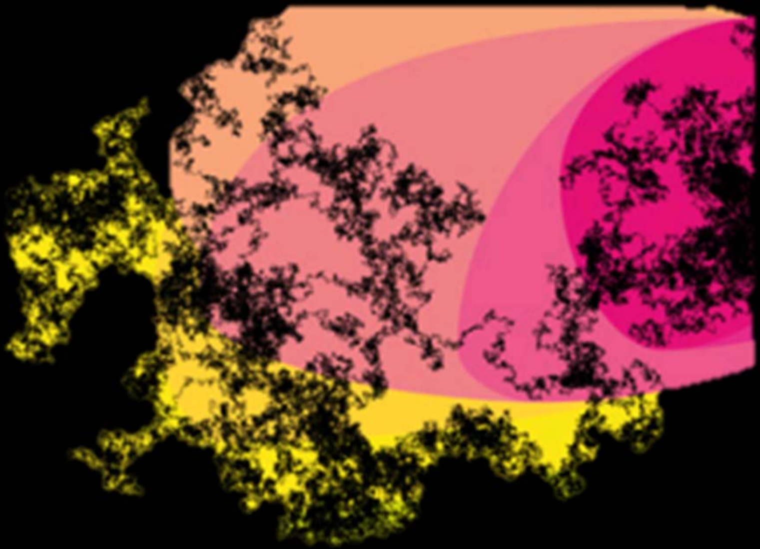
- PageRank

- Betweenness
- Local-Sphere of Influence
- …



Dynamic Processes over Networks



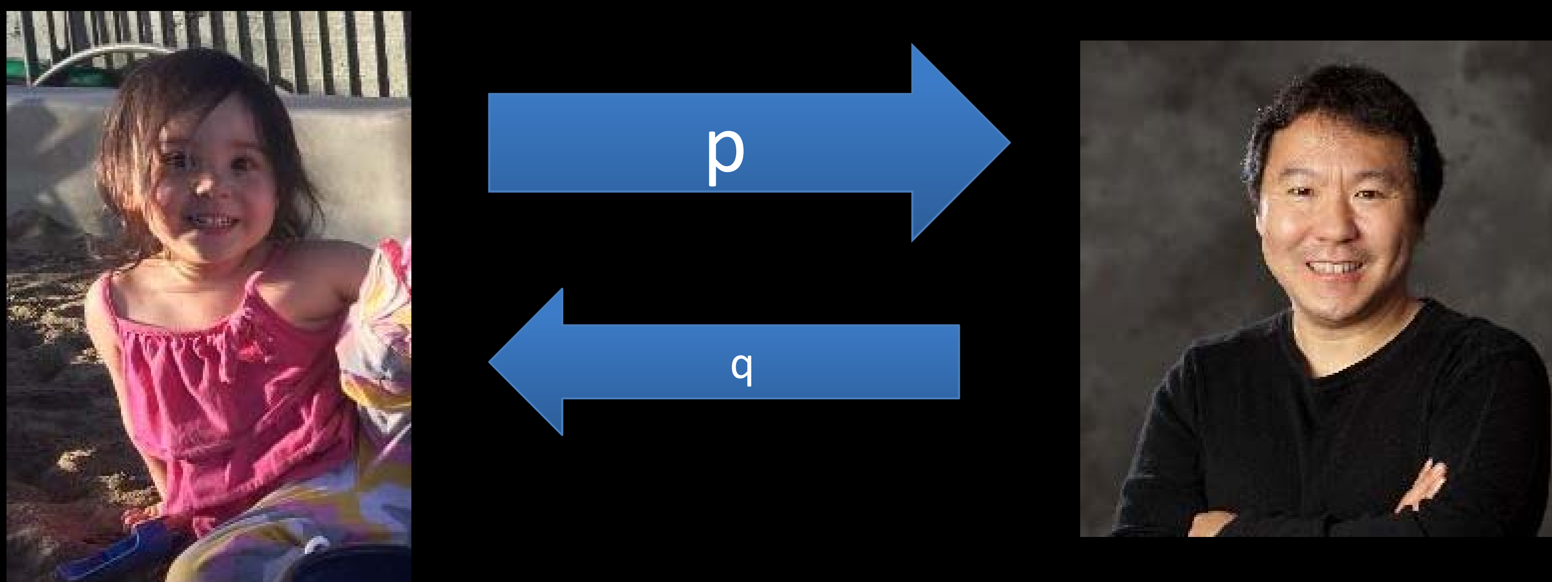


Impact of Influence Dynamics on Network Centrality?

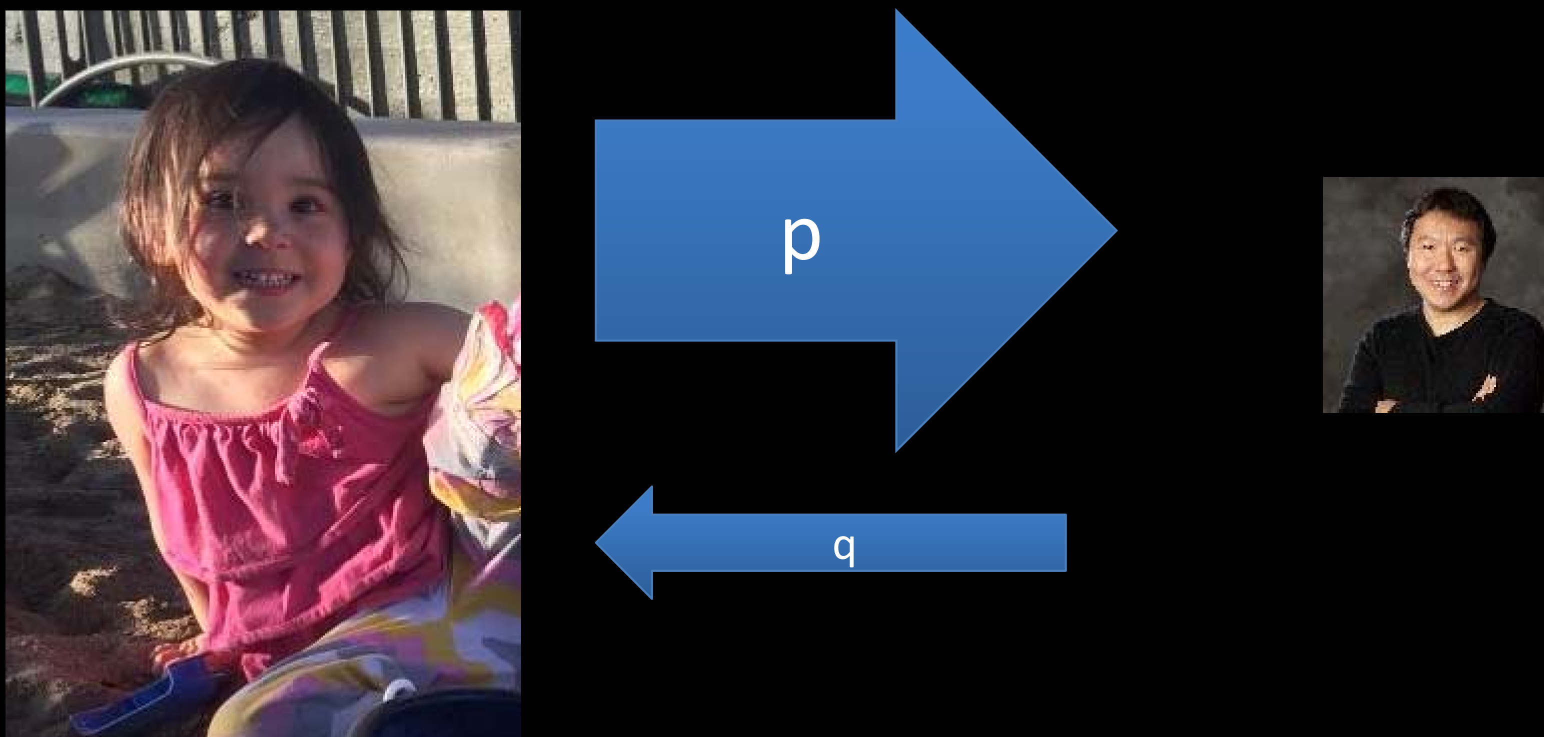
- Influence Process: *D*
- Social Network: *G* = (*V*, *E*)

Static measures may not sufficiently capture social-influence centrality

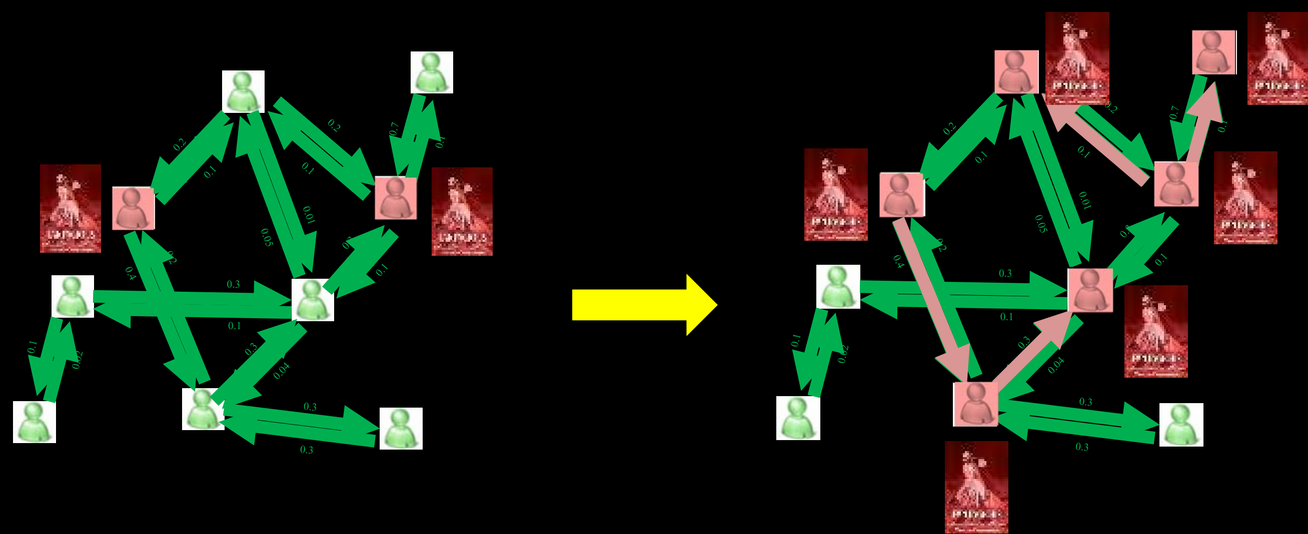
Two-Node Network Influence



Two-Node Network Influence



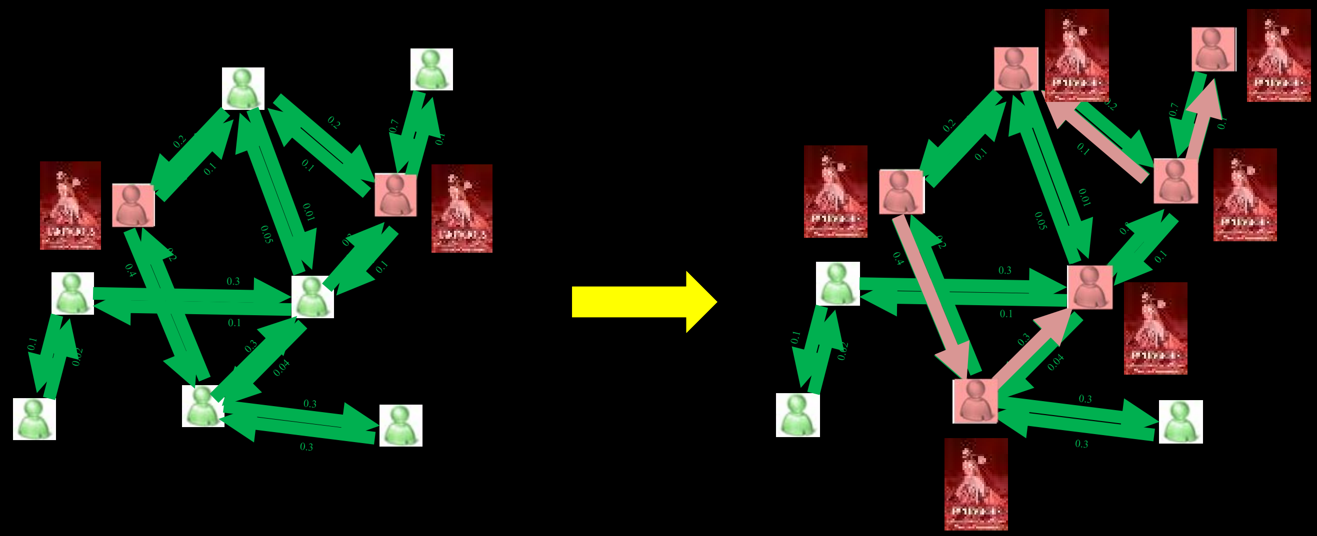
The Underlying Interplay



- Probabilistic View: Powerset Networks

$$P_{G,D}\,[S,T]$$

The Underlying Interplay



- Probabilistic View: Powerset Networks

$$P_{G,D}\,[S,T]$$

- Utility View: The Influence Spread (KKT)

$$\sigma_{G,D}\,(S) = \sum \left(\,|\mathrm{T}|\; P_{G,D}\,[S,T]\, \right)$$

Game Theoretical View of Social Influence

Social-Influence Cooperative Games:

$$\sigma_{G,D}(S)$$

Shapley Values

$$\mathbf{SV}_{\sigma}(k) = E_{\pi}[\,\sigma(S_{\pi,k} + k) - \sigma(S_{\pi,k})\,]$$

$S_{\pi,k}$: *players placed before k according to* π

Expected Marginal Contribution

Shapley's Axiomatic Characterization

$$\phi: \mathbb{R}^{2^N} \to \mathbb{R}$$

- **Efficiency**
- **Symmetry**
- **Linearity**
- **Null Player**

A Game-Theoretical Approach

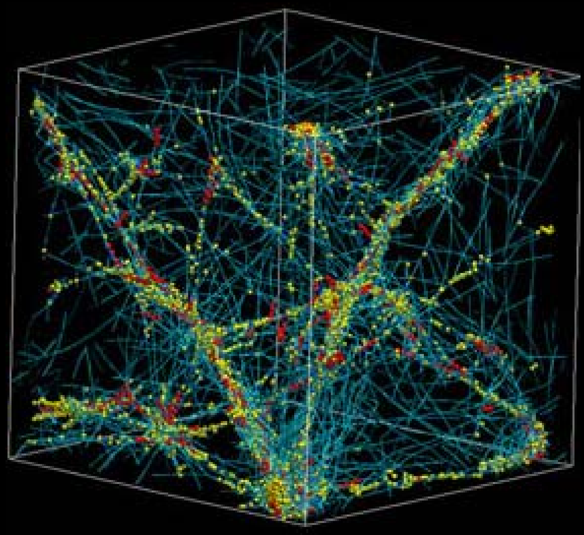
the impact of an influence process on network centrality:

- Social-Influence Games:

$$\sigma_{G,D}(S)$$

- Shapley Centrality:

$$[\, sv_{\sigma}(v) \,]_v$$



Dimension-Reduction of Network Data

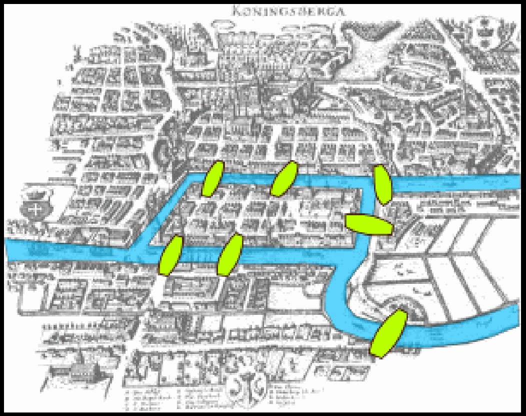
Probabilistic Model

Utility Model

Centrality

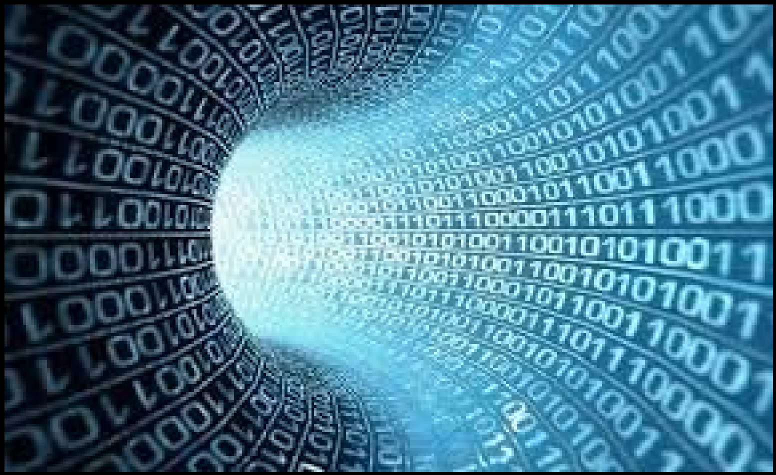
What does the Shapley centrality capture?

Fantastic Research Problem



- Graph Theory (Euler, circa 1736)
- Social Influence (1950s, then circ 2002)
- Cooperative Game Theory (1950s)

Network Science in the Age of Big Data



- Mathematically Meaningful
- Algorithmically Scalable
- Experimentally Validatable

Mathematical Question

What does the Shapley value of the cooperative social-influence game reflect?

Mathematical Question

What does the Shapley value of the cooperative social-influence game reflect?

Axiomatic Characterization

Motivated by:

1. Altman and Tennenholtz: PageRank Axioms
2. Palacios-Huerta and Volij: Intellectual Influence
3. Shapley's Axioms

Representation Theorem

Soundness:

– *Social-influence Shapley centrality satisfies Axioms 1-6*

Completeness:

– ***The solution to Axioms 1-6 is unique***

Influence-Centrality Axioms

1. ***Anonymity:*** *invariant under permutation*

Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. ***Normalization: average centrality is 1***

Influence-Centrality Axioms

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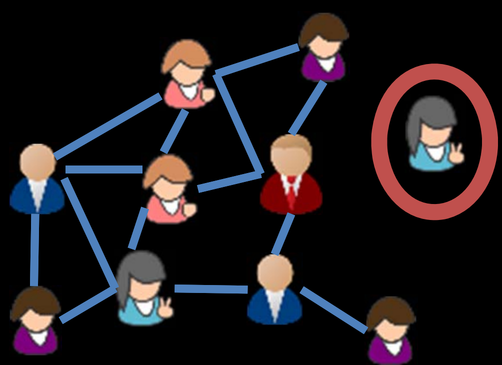


Isolated Nodes

$$P_{G,D}[S+u,T+u] = P_{G,D}[S,T]$$

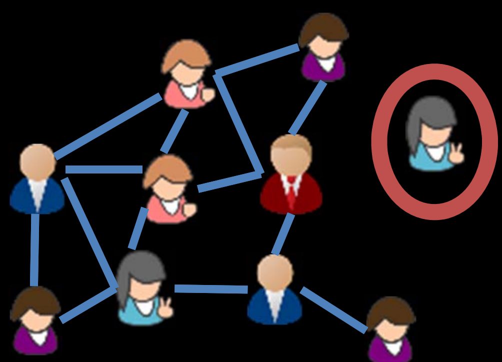
$$\Downarrow$$

$$P_{G,D}[u,u] = P_{G,D}[\ ,\ ] = 1$$

$$P_{G,D}[S,\mathrm{T}+u] = 0$$

Influence-Centrality Axioms

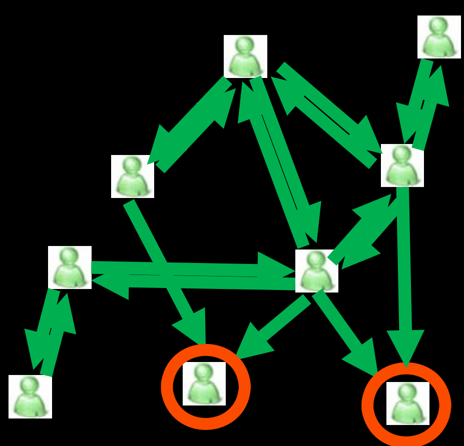
1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. ***Isolated Nodes: centrality of isolated is 1***



Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Nodes: centrality of isolated is 1

Sink Node



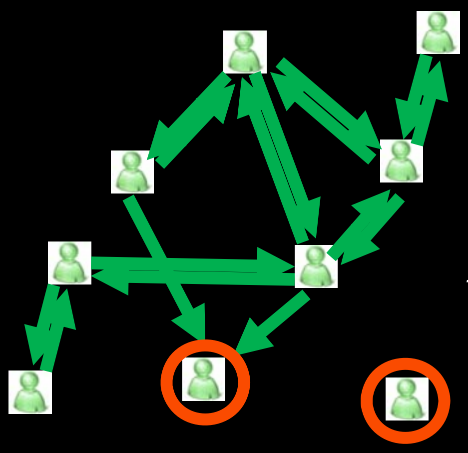
$$P_{G,D}[S+u,T+u] = P_{G,D}[S,T] + P_{G,D}[S,T+u]$$

$$\Downarrow$$

$$P_{G,D}[u,u] = P_{G,D}[\ ,\ ] = 1$$

Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Nodes: centrality of isolated is 1

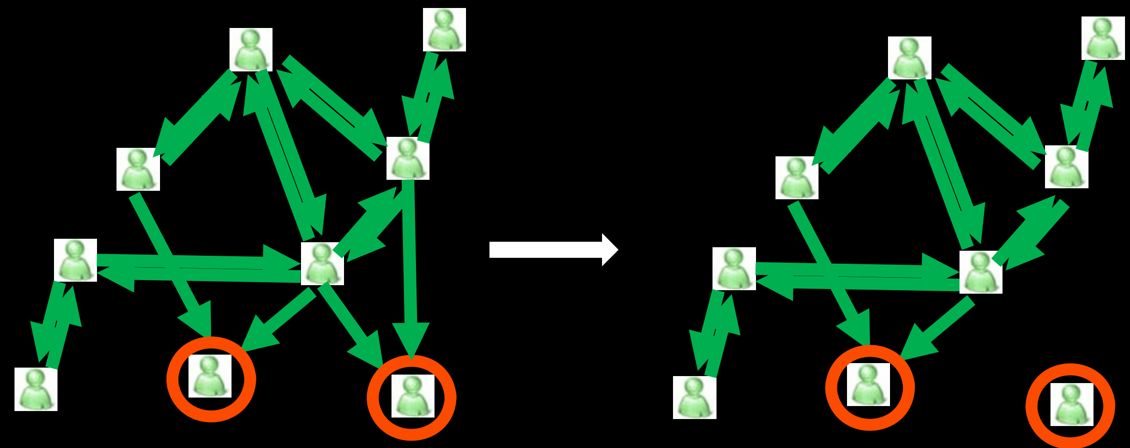


Projection of a Sink Node

$$P_{G\backslash u,D}[S,T] := P_{G,D}[S,T] + P_{G,D}[S,T+u]$$

Influence-Centrality Axioms

1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Node: centrality of isolated is 1
4. ***Independence of Sink Nodes: sink-node projection preserves centrality of other sink nodes***



Bayesian Social Influence

- Social network: $G = (V, E)$
- Influence Model:
  - Processes: $D[1] \ldots D[r]$
  - A prior distribution: $\boldsymbol{\lambda} = (\lambda[1] \ldots \lambda[r])$,

$$P_{G,D}[S,T] = \sum \lambda[\theta]\ P_{G,D[\theta]}[S,T]$$

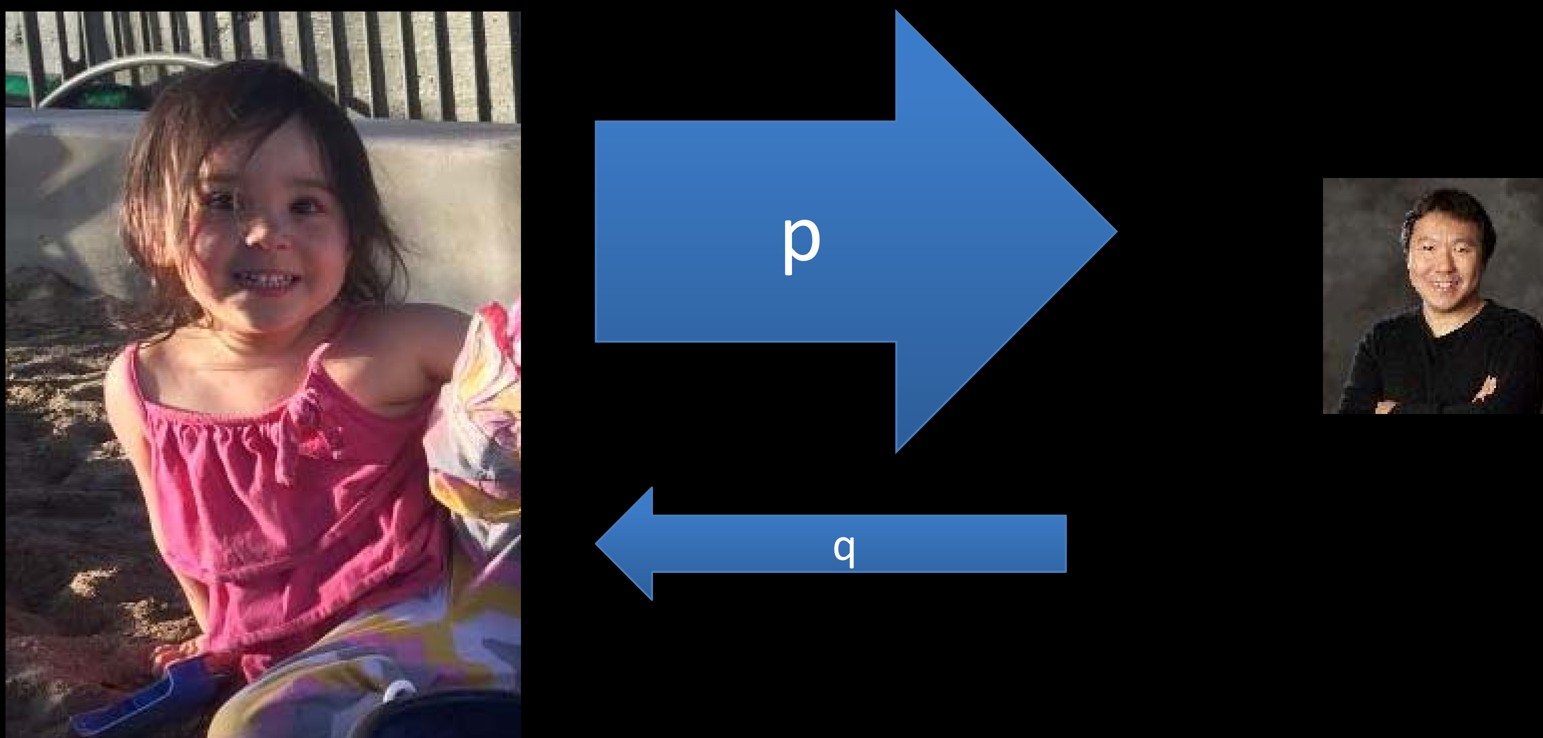
Axiom 5: Bayesian

- Social network: $G = (V, E)$
- Influence Model:
  - Processes: $D[1] \ldots D[r]$
  - A prior distribution: $\boldsymbol{\lambda} = (\lambda[1] \ldots \lambda[r])$,

$$P_{G,D}[S,T] = \sum \lambda[\theta]\ P_{G,D[\theta]}[S,T]$$

5. Bayesian: ***social-influence centrality satisfies the linearity-of-expectation principle***

Two-Node Network Influence



Likely Parent-Child Influence Model

1

0

Nash Bargaining

Nash Bargaining

1

0

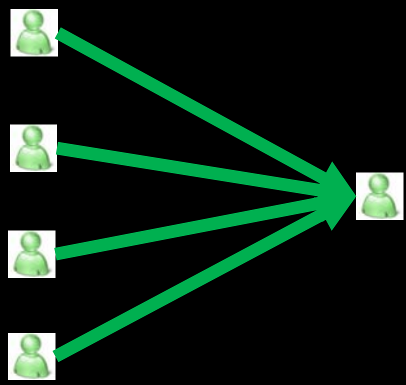
3/2

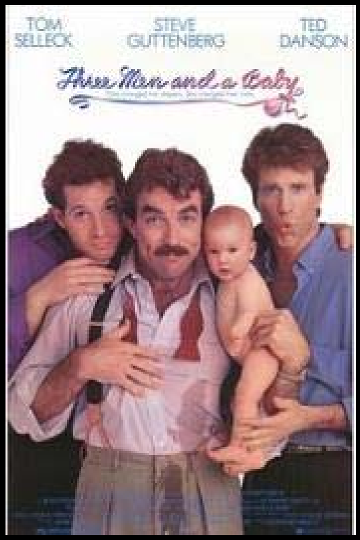
1/2

How to Improve Centrality in a Family?

Two Tiger Parents?

Critical Set Instances





- *R, v*
  1. *Pr[R,R+v] = Pr [R+v,R+v] = 1*
  2. *Pr [S,S] = 1*

Bargaining with Two Tiger Parents

Bargaining with Two Tiger Parents

Axiom 6: Bargaining with Critical Sets

***6. Bargaining with Critical Sets:** the centrality of v is r/(r+1)*

Influence-Centrality Axioms

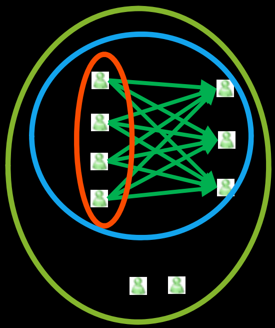
1. Anonymity: invariant under permutation
2. Normalization: average centrality is 1
3. Isolated Node: centrality of isolated is 1
4. Independence of Sink Nodes: sink-node projection preserves centrality of other sink nodes
5. Bayesian: social-influence centrality satisfies linearity-of-expectation principle
6. Bargaining with Critical Sets: the centrality of v is r/(r+1)

Axiomatic Characterization

The social-influence Shapley centrality is the unique centrality measure that satisfies Axioms 1-6.

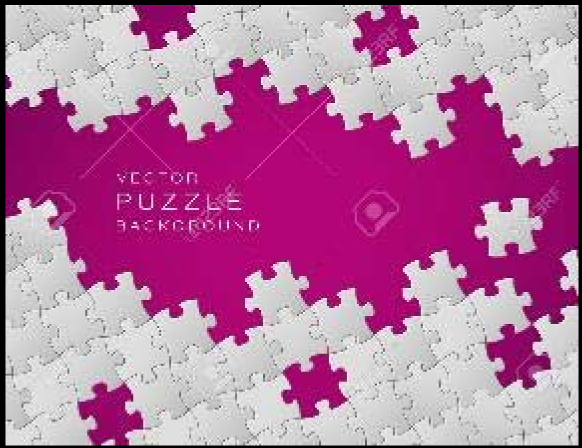
Our Proof: Simplicity

- Following Myerson's proof strategy



- Vector Space: $\{ P_{G,D} [S,T] \}_{\mathrm{S,T}}$ (the probability profile)
- A Full-Rank Basis: the critical set instances and extensions
- Linear Maps: axiom-conforming centrality measures
- Uniqueness: for critical set instances and their extensions

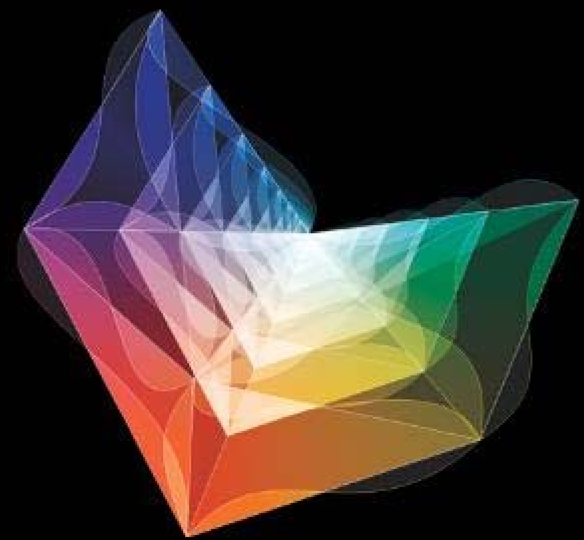
Our Proof: Complexity



- More cares than Myerson's proof of Shapley's theorem
- Our axiomatic framework is based on the *influence model*, rather than on *influence spread*
- The probabilistic profile has *higher dimensionality* than the influence-spread profile

The Space of Social Influences

Dimensions:

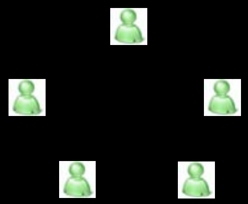


the number of pairs (*S*,*T*) satisfying

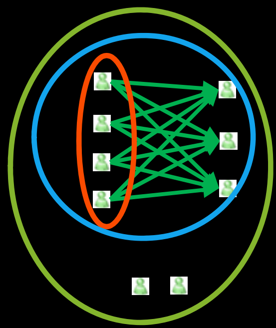
1. S ⊂ T ⊆ V, and
2. S not in {∅,V}

The Space of Social Influences

- Null Instances:



- Basis Instances:



- General Instances:

Implied Properties

- Nondiscrimination (Symmetry) Property:
  - Nodes with same marginal influence-spread profile have the same Shapley centrality
- Independence of Irrelevant Alternatives
  - Disconnected influence components define their own Shapley centrality

Dimension-Reduction of Network Data

Probabilistic Model

Utility Model

Centrality

- Axiomatic analysis of dimension reduction
- Comparative framework

An Empirically Observed Theorem

Symmetric Independent Cascade Model

- G *undirected*
- $p_{uv} = p_{vu}$

Shapley Symmetry of the Symmetric IC Model:

The Shapley centrality of each node is 1

- *Undirected* live graph
- Principle of *deferred decision*

Shapley Symmetry of the Symmetric IC Model

At first glance: surprising and counterintuitive

- limitation of the Shapley centrality?
  - independent of both network structure and symmetric IC edge probabilities.

- limitation of the symmetric IC model?
  - The "pair-wise symmetry and independence" condition is an extreme assumption (that rarely holds for real-world influence propagation).

Sheds Light on both Network Influence and Game-Theoretical Centrality

The Shapley centrality remarkably reveals this symmetry because:

- instead of measuring individual influence spreads in isolation from other nodes
- captures the expected “irreplaceable power” of each node in group influence
- for the symmetric IC model, the equal Shapley centrality exactly points out that all nodes in the network are replaceable if their are equally positioned in a random order

Two Categories of Axioms

- **Principle Axioms:**
  - Anonymity
  - Bayesian
- **Choice Axioms:**
  - Normalization
  - Isolated Node
  - Independence of Sink Nodes
  - Bargaining with Critical Sets

Two Categories of Axioms

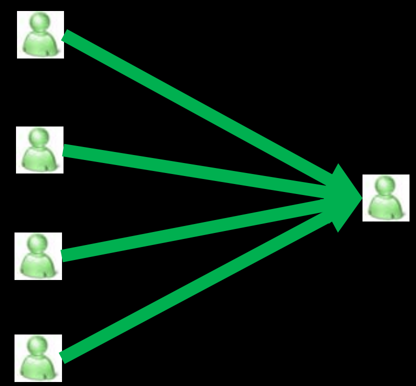
- **Principle Axioms:**

  the essence of common desirable properties

- **Choice Axioms:**

  succinctly distill the comparative differences between different formulations.

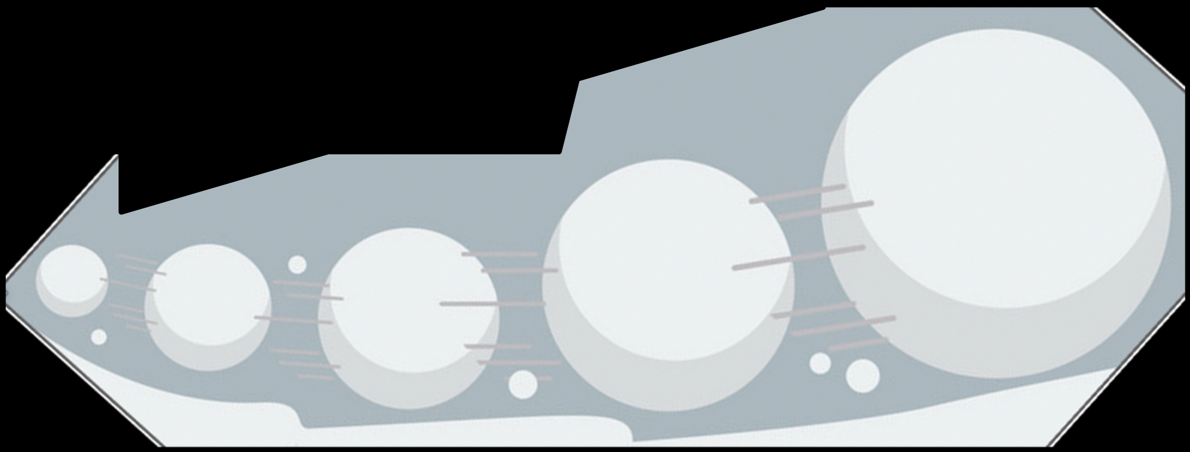
Deterministic Basis for Stochastic Influence



Critical Sets: Many to One Influence

Richer Influence Models

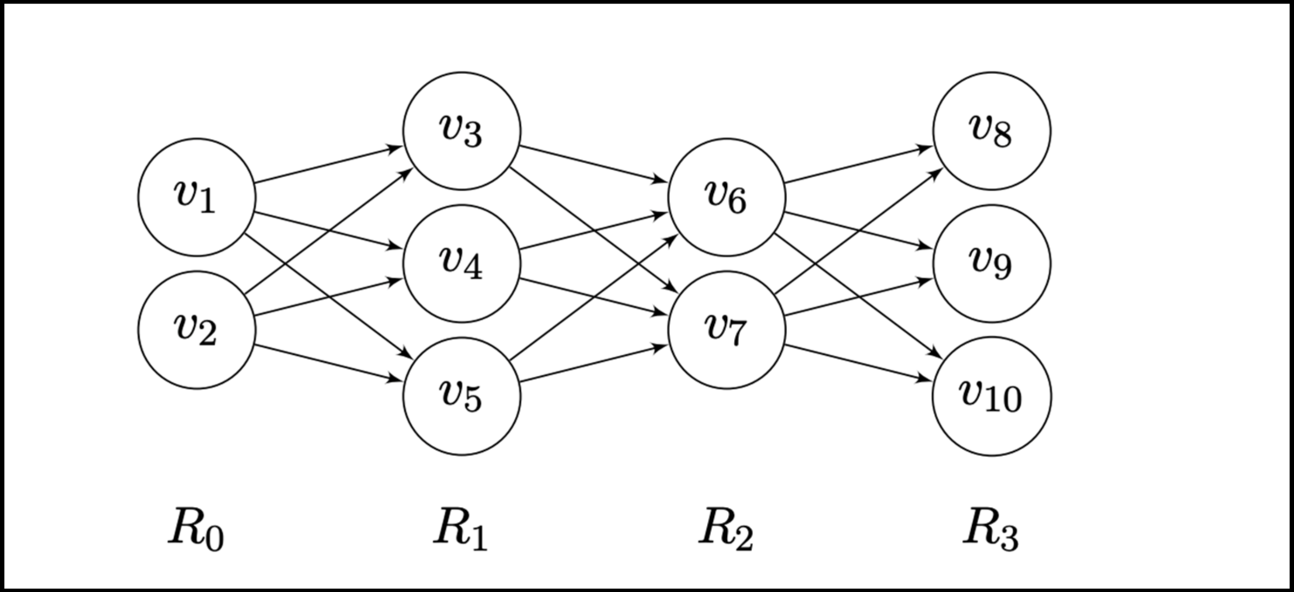
- **Cascading Sequences**



- **Influencing:** Stochastic Cascading Profiles

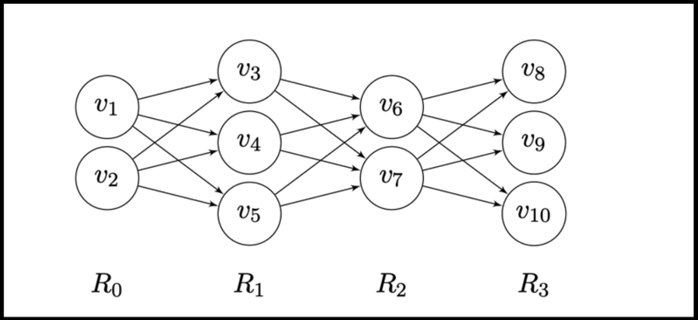
BFS-Propagation

- **Networking Broadcasting**



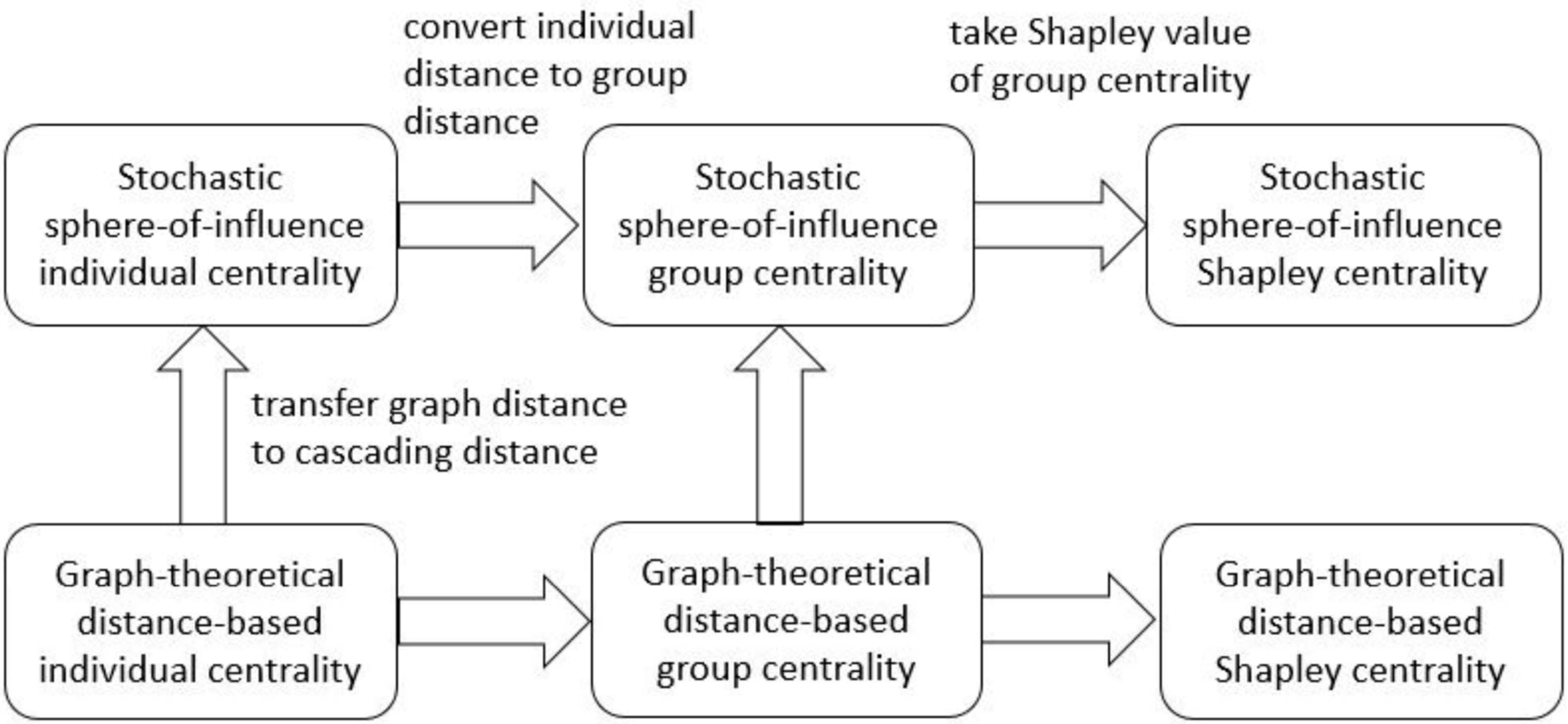
Graph-Theoretical Basis of Influence Models

Theorem: BFS Propagation Profiles form a basis of all stochastic cascading profiles.

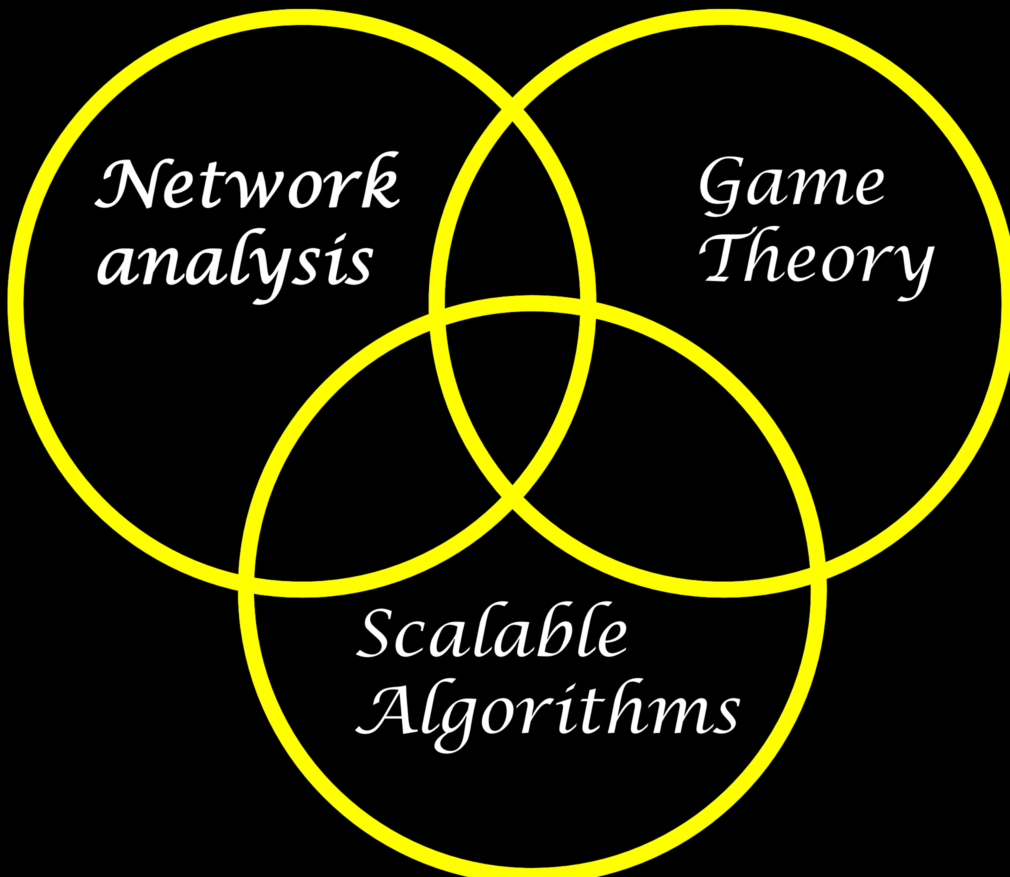


Axiomatic Characterization: Under principle Anonymity and Bayesian axioms, the centrality formulation is uniquely characterized by the centrality formulation of layered graphs.

A Systematic Road Map



Road map for the systematic extension of graph-theoretical distance-based centralities to influence-based centralities.


Network analysis
Game Theory
Scalable Algorithms

Interplay Between Dynamic Processes and Network Structures

Shapley centrality:

- Axiomatically characterized by
  - permutation invariance, scaling invariance, Bayesian linearity
  - three simple boundary cases
- Efficient to approximate
- Extensions:
  - Weighted influence models
    - node has different weights, both algorithm and axiomatization can be extended
  - Axiomatization based on influence spread

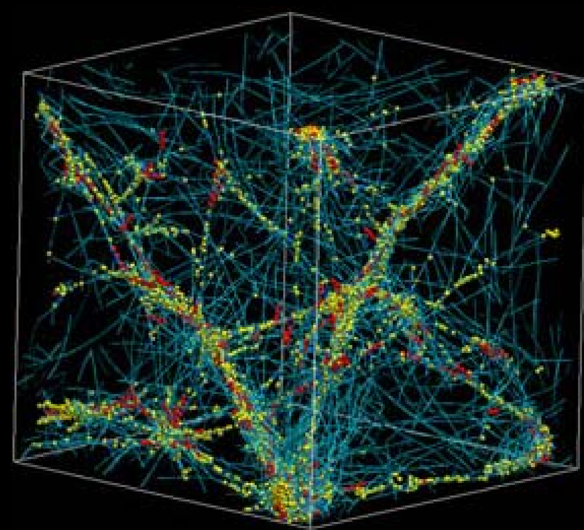
Future Directions

Broader and deeper understanding of game-theoretical approach to network analysis

- Impact of network dynamics on clusterability
- Community identification
- Bounded rationality

Comprehensive/comparative algorithmic and mathematical framework for network analysis

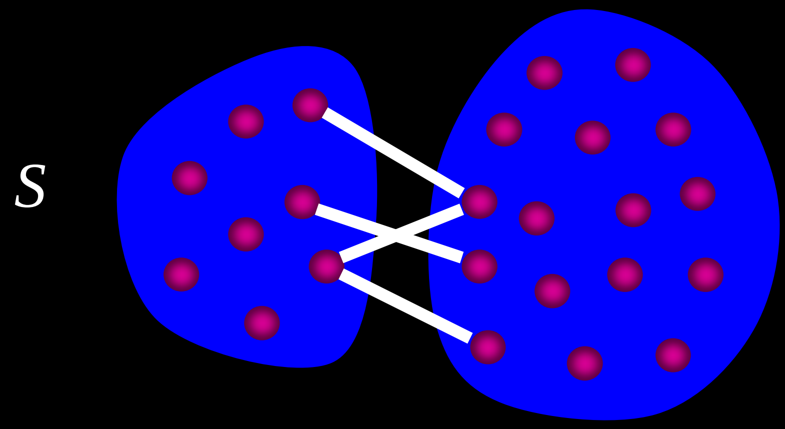
Interplay Between Dynamic Process and Network Structures



What is the impact on network concepts?

Clusterability and Community Characterization

- Conductance
- Cut-ratio



- Modularity
- PageRank Modularity

Holy Grail of Network Science

To understand the *network essence*
that underlies the observed
sparse-and-multifaceted network data

Thank You!