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Optimal Power Control in Rayleigh-fading Heterogeneous Networks

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***Abstract*—Heterogeneous wireless networks employ varying degrees of network coverage using power control in a multi-tier configuration, where low-power femtocells are used to enhance performance, e.g., optimize outage probability. We study the worst outage probability problem under Rayleigh fading. As a by-product, we solve an open problem of convergence for a previously proposed algorithm in the interference-limited case. We then address a total power minimization problem with outage specification constraints and its feasibility condition. We propose a dynamic algorithm that adapts the outage probability specification in a heterogeneous network to minimize the total energy consumption and simultaneously guarantees all the femtocell users a min-max fairness in terms of the worst outage probability.**

Index Terms— **Optimization, nonnegative matrix theory, outage probability, power control, femtocell networks.**

I. Introduction

Conventional wireless cellular networks are designed to provide network coverage over large areas and support many users. Most recently, 3GPP LTE-advanced investigated heterogeneous network deployments to improve system performance as well as effectively enhance network coverage, especially for in-building coverage [1], [2]. Heterogeneous networks use a mix of higher-tier macrocells to extend network reach and lower-tier femtocells to enhance performance, e.g., optimize outage probability, within the same frequency band [1], [2]. However, interference management is required to ensure that the heterogeneous users control their transmit power to mitigate performance loss due to mutual interference. To enhance decentralized deployment, users also need to adapt their transmit power with minimal signaling overhead [1], [2].

Outage probability is an important performance parameter for reliable wireless communication. A link outage is declared when the received Signal-to-Interference-and-Noise Ratio (SINR) falls below a given threshold that is often computed from the quality of service requirement. The statistics of the SINR, and consequently that of the outage probability, are also dependent on other network parameters, including imperfections due to the channel statistical fading, the additive background noise, user mobility and the dynamics of power control updates. In this paper, we focus on the short-term channel fading, which can be modeled by either a Rayleigh, a Ricean or a Nakagami distribution depending on the environment under consideration [3]. In particular, we focus on Rayleigh fading that is relevant to in-building coverage model and heavily built-up urban environments [1], [3].

In [4], the authors proposed tracking Rayleigh-fading fluctuation to reduce outage in a macrocell network. In [5], algorithms that adapt SINR requirements for utility maximization were studied for a macro-femtocell network. The authors in [6] studied subcarrier and power control to minimize outage in an OFDMA cellular network. The authors in [7] studied the worst outage probability problem and the total power minimization problem in interference-limited Rayleigh-fading networks. The authors in [8] proposed an iterative algorithm for the total power minimization problem, but left its feasibility issue open, which we address in this paper using the nonlinear Perron-Frobenius theory in [9], [10], [11] and nonnegative matrix theory [12]. Interestingly, we also show that the worst outage probability problem, similar to its certainty-equivalent margin (CEM) counterpart (also known as the max-min weighted SINR problem), is a *nonlinear* eigenvalue problem, whose optimal value and optimal solution are associated with a Perron eigenvalue and eigenvector, respectively.

It is imperative to design heterogeneous networks that are capable of *adapting* their behavior to be both spectral and energy efficient [1]. For this purpose, we propose a dynamic power control algorithm that allows each femtocell user to adapt its outage probability specification to minimize the total energy consumption in the system and guarantees a min-max fairness in terms of worst outage probability to all the femtocell users. This allows users to adapt their performance quality in a dynamic environment without the need of a centralized admission control mechanism (which is often not practically feasible due to the ad-hoc architecture).

Overall, the contributions of the paper are as follows:

1) We solve analytically the worst outage probability minimization problem subject to a total power constraint (for downlink scenario) or individual power constraints (for uplink scenario). We propose a fast algorithm to compute the solution. As a by-product, we solve an open problem of convergence for a previously proposed algorithm in [7] for the interference-limited case.
2) We establish a tight relationship between the worst outage probability problem and its certainty-equivalent margin counterpart, and utilize the connection to find useful bounds and the convergence rate of our algorithm.
3) We characterize analytically the feasibility condition of the total power minimization problem with both outage specification and individual power constraints, and provide feasibility bounds based on the problem parameters, e.g., SINR thresholds, the outage thresholds.
4) We propose a dynamic algorithm for the graceful handling of infeasibility in a femto-macro cell network.

The algorithm optimizes the overall energy consumption based on adaptive outage probability specification that guarantees the worst outage probability to all the users.

This paper is organized as follows. We introduce the system model in Section II. In Section III, we solve the problem of minimizing the worst outage probability analytically and propose a fast algorithm. In Section IV, we study the feasibility condition of a total power minimization problem, and propose a dynamic power control algorithm that adapts the outage probability specification to minimize the total energy. In Section V, we illustrate the numerical performance of our algorithms. We conclude the paper in Section VI.

The following notation is used. Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors, italics denote scalars, and $\mathbf{u} \geq \mathbf{v}$ denotes componentwise inequality between vectors $\mathbf{u}$ and $\mathbf{v}$. We also let $(\mathbf{B}\mathbf{y})_l$ denote the lth element of $\mathbf{B}\mathbf{y}$. Let $\mathbf{x}/\mathbf{y}$ denote the vector $[x_1/y_1, \ldots, x_L/y_L]^\top$. We write $\mathbf{B} \geq \mathbf{F}$ if $B_{ij} \geq F_{ij}$ for all i, j. The Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted as $\rho(\mathbf{F})$, and the Perron (right) and left eigenvector of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F}) \geq \mathbf{0}$ and $\mathbf{y}(\mathbf{F}) \geq \mathbf{0}$ (or, simply $\mathbf{x}$ and $\mathbf{y}$, when the context is clear) respectively. Recall that the Perron-Frobenius eigenvalue of $\mathbf{F}$ is the eigenvalue with the largest absolute value. Assume that $\mathbf{F}$ is an irreducible nonnegative matrix. Then $\rho(\mathbf{F})$ is simple and positive, and $\mathbf{x}(\mathbf{F}), \mathbf{y}(\mathbf{F}) > \mathbf{0}$ [13]. The super-script $(\cdot)^\top$ denotes transpose. We denote $\mathbf{e}_l$ as the lth unit coordinate vector and $\mathbf{I}$ as the identity matrix. For any vector $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_L)^\top \in \mathbb{R}^L$, let $e^{\tilde{\boldsymbol{\gamma}}} = (e^{\tilde{\gamma}_1}, \ldots, e^{\tilde{\gamma}_L})^\top$.

II. System Model

Consider a multiuser communication system with L users (logical transmitter/receiver pairs) sharing a common frequency. Each user employs a singe-user decoder, i.e., treating interference as additive Gaussian noise, and has perfect channel state information at the receiver. Our system with L users can be modeled by a Gaussian interference channel having the baseband signal model $\boldsymbol{y}_l = \boldsymbol{h}_{ll}\boldsymbol{x}_l + \sum_{j \neq l} \boldsymbol{h}_{lj}\boldsymbol{x}_j + \boldsymbol{z}_l$, where $\boldsymbol{y}_l \in \mathbb{C}^{1\times 1}$ is the received signal of the lth user, $\boldsymbol{h}_{lj} \in \mathbb{C}^{1\times 1}$ is the channel coefficient between the transmitter of the jth user and the receiver of the lth user, $\boldsymbol{x} \in \mathbb{C}^{N\times 1}$ is the transmitted (information carrying) signal vector, and $\boldsymbol{z}_l$'s are the i.i.d. additive complex Gaussian noise coefficient with variance $n_l/2$ on each of its real and imaginary components (thus n_l being the noise power at the lth receiver). At each transmitter, the signal is constrained by an average power constraint, i.e., $\mathbb{E}[|\boldsymbol{x}_l|^2] = p_l$, which we assume to be upper bounded by $\bar{p}$ for all l. The vector $(p_1, \ldots, p_L)^\top$ is the transmit power vector, which is the optimization variable of interest in this paper.

Under Rayleigh fading, the power received from the jth transmitter at lth receiver is given by $G_{lj}R_{lj}p_j$ where G_{lj} represents the nonnegative path gain between the jth transmitter and the lth receiver (it may also encompass antenna gain and coding gain) that is often modeled as proportional to $d_{lj}^{-\gamma}$, where d_{lj} denotes distance, γ is the power fall-off factor, and R_{lj} models Rayleigh fading and are independent and exponentially distributed with unit mean. The distribution of the received power from the jth transmitter at the l receiver is then exponential with mean value $E[G_{lj}R_{lj}p_j] = G_{lj}p_j$.

Next, we define the nonnegative matrix $\mathbf{F}$ with the entries:

$$F_{lj} = \begin{cases} 0, & \text{if } l = j \\ \frac{G_{lj}}{G_{ll}}, & \text{if } l \neq j \end{cases} \tag{1}$$

and

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top. \tag{2}$$

Moreover, we assume that $\mathbf{F}$ is irreducible, i.e., each link has at least an interferer.

We assume that the SINR for the lth receiver, e.g., a linear matched-filter receiver, is given by [7]:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{R_{ll}p_l}{\sum_{j \neq l} F_{lj}R_{lj}p_j + v_l}. \tag{3}$$

Now, (3) is a random variable that depends on Rayleigh fading. The transmission from the lth transmitter to its receiver is successful if $\mathsf{SINR}_l(\mathbf{p}) \geq \beta_l$ (zero-outage), where β_l is a given threshold for reliable communication. An outage occurs at the lth receiver when $\mathsf{SINR}_l(\mathbf{p}) < \beta_l$, and we denote this outage probability by $P(\mathsf{SINR}_l(\mathbf{p}) < \beta_l)$.

Power constraint is a critical design parameter in a wireless network [3]. We focus on the power constraint $\mathcal{P}$ being either a total power constraint or individual power constraints, i.e.,

$$\mathcal{P} = \{\mathbf{p} \mid \mathbf{1}^\top \mathbf{p} \leq \bar{P}\} \quad \text{or} \quad \mathcal{P} = \{\mathbf{p} \mid p_l \leq \bar{p} \ \ \forall l\}. \tag{4}$$

III. Worst Outage Probability Minimization

The problem of minimizing the worst outage probability can be formulated as

$$\begin{array}{ll} \text{minimize} & \max_l \ P(\mathsf{SINR}_l(\mathbf{p}) < \beta_l) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{5}$$

Assuming independent Rayleigh fading at all signals, the outage probability of the lth user can be given analytically by [7]:[1]

$$P(\mathsf{SINR}_l(\mathbf{p}) < \beta_l) = 1 - e^{\frac{-v_l\beta_l}{p_l}} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1}. \tag{6}$$

In the following, we denote $O_l(\mathbf{p}) = 1 - e^{-v_l\beta_l/p_l} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1}$. Observe that the probability of successful transmission, i.e., the complement of (6), is simply the product of two factors, namely, $e^{-v_l\beta_l/p_l}$ and $\prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1}$, which are the probability of successful transmission in a noise-limited Rayleigh-fading channel and an interference-limited Rayleigh-fading channel respectively.

[1] A closed form expression was first derived in [14], but we use another equivalent form derived in [7].

Using (6), (5) simplifies to a deterministic problem:

$$\begin{array}{ll} \text{minimize} & \max_{l} \;\; O_l(\mathbf{p}) = 1 - e^{\frac{-v_l\beta_l}{p_l}} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1} \\ \text{subject to} & \mathbf{p} \in \mathcal{P}. \end{array} \tag{7}$$

Note that (7) is always feasible and its optimal solution is strictly positive. Previous work in the literature, e.g., [7], only considered (7) for the interference-limited case, i.e., $\mathbf{v} = 0$ and *without any power constraint*. In this special case, [7] showed that (7) can be reformulated as a geometric program (GP), and be solved efficiently using the interior point method [15].

In the following, we give a reformulation of (7) as a convex optimization problem (not a convex GP formulation in general, but reduces to one in the interference-limited special case). By exploiting the nonlinear Perron-Frobenius theorem, we propose a fast algorithm[2] (no configuration whatsoever and orders of magnitude faster than standard convex optimization algorithms, e.g., interior-point method) to solve (7) optimally. As a by-product, it resolves an open problem on the convergence of a previously proposed heuristic algorithm in [7]. Furthermore, we derive analytically the optimal value and solution of (7) in terms of a Perron eigenvalue and eigenvector of a specially constructed nonnegative matrix respectively.

We next introduce the auxiliary variable τ and write (7) in the epigraph form (augmenting the constraint set with an additional L constraints):

$$\begin{array}{ll} \text{minimize} & \tau \\ \text{subject to} & 1 - e^{\frac{-v_l\beta_l}{p_l}} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1} \leq \tau \;\; \forall\, l, \\ & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p},\; \tau. \end{array} \tag{8}$$

By letting $\alpha = -\log(1-\tau)$ and rewriting the L augmented constraints in (8), (8) is equivalent to the following problem:

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & v_l\beta_l/p_l + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right) \leq \alpha \;\; \forall\, l, \\ & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p},\; \alpha. \end{array} \tag{9}$$

We call the first L constraints of (9) the *outage constraints*, and denote the optimal solution to (9) by $(\mathbf{p}^\star, \alpha^\star)$. Now, (9) is nonconvex in $(\mathbf{p}, \alpha)$. However, by making a logarithmic change of variable in $\mathbf{p}$, i.e., $\tilde{p}_l = \log p_l$ for all l, (9) can be converted into the following convex optimization problem in $(\tilde{\mathbf{p}}, \alpha)$:[3]

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & v_l\beta_l e^{-\tilde{p}_l} + \sum_j \log\left(1 + \beta_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}\right) \leq \alpha \;\; \forall\, l, \\ & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}},\; \alpha. \end{array} \tag{10}$$

[2]Some key computational considerations are the extremely fast signal processing requirement at the transceiver chip and the decentralized environment.

[3]Note that (9) cannot be rewritten as a standard GP formulation, as has been done in [7]. Nevertheless, after a logarithmic change of variables, a convex form can still be obtained as shown here.

Though solving the nonconvex problem (9) is equivalent to solving the convex problem (10), we next use a nonlinear Perron-Frobenius theory-based approach to solve (9) optimally. Using nonnegative matrix theory, we then connect (9) to the Lagrange duality of (10) (cf. Lemma 2 later).

Lemma 1: At optimality of (9), the outage constraints in (9) are tight:

$$v_l\beta_l/p_l^\star + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j^\star}{p_l^\star}\right) = \alpha^\star \;\; \forall\, l. \tag{11}$$

Furthermore, if $\mathcal{P} = \{\mathbf{p} \mid \mathbf{1}^\top \mathbf{p} \leq \bar{P}\}$, we have $\mathbf{1}^\top \mathbf{p}^\star = \bar{P}$, and if $\mathcal{P} = \{\mathbf{p} \mid p_l \leq \bar{p} \,\forall\, l\}$, we have $p_i^\star = \bar{p}$ for some i.

Proof: First, we note that it has been pointed out in [7] that all the outage constraints are tight for the interference-limited case, i.e., $\mathbf{v} = \mathbf{0}$. We prove the first part of Lemma 1 for the general case here. Clearly, the function on the lefthand side of the lth outage constraint in (9) is monotone increasing in p_j, $j \neq l$, and monotone decreasing in p_l. Suppose the lth constraint is not tight at optimality, i.e., $v_l\beta_l/p_l^\star + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j^\star}{p_l^\star}\right) < \alpha^\star$. Then, we choose a feasible power $p_l < p_l^\star$ such that the evaluated value of $v_l\beta_l/p_l + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j^\star}{p_l}\right)$ is still less than $\alpha^\star$. Now, $v_j\beta_j/p_j^\star + \sum_{k\neq l} \log\left(1 + \frac{\beta_j F_{jk} p_k^\star}{p_j^\star}\right) + \log\left(1 + \frac{\beta_j F_{jl} p_l}{p_j^\star}\right)$ for all $j \neq l$. This implies that the value of α can be further decreased, i.e., $\alpha < \alpha^\star$, which contradicts the assumption. Hence, the lth constraint must be tight at optimality for all l.

We next prove the second part for $\mathcal{P} = \{\mathbf{p} \mid p_l \leq \bar{p} \,\forall\, l\}$. Suppose $p_l^\star < \bar{p}$ at optimality for all l. Let a positive scalar $a = \min_l \bar{p}/p_l^\star > 1$, and choose a feasible power $\mathbf{p} = a\mathbf{p}^\star$, which evaluates the outage constraints as $v_l\beta_l/p_l + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right) = v_l\beta_l/ap_l^\star + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j^\star}{p_l^\star}\right) < v_l\beta_l/p_l^\star + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j^\star}{p_l^\star}\right) = \alpha^\star$ for all l. This implies that α can be further decreased, i.e., $\alpha < \alpha^\star$, which contradicts the assumption. Hence, $p_i^\star = \bar{p}$ for some i. A similar proof can be given when $\mathcal{P} = \{\mathbf{p} \mid \mathbf{1}^\top \mathbf{p} \leq \bar{P}\}$ and is omitted. ∎

Remark 1: We give an analytical solution to (11) and thus the optimal solution of (7) in Section III-C (see Table I later).

By exploiting a connection between the nonlinear Perron-Frobenius theory in [9], [10] and the algebraic structure of (9), we propose the following algorithm (with geometric convergence rate and no configuration whatsoever) that computes the optimal solution of (9). We let k index discrete time slots.

Algorithm 1 (Worst Outage Probability Minimization):

1) Update power $\mathbf{p}(k+1)$:

$$p_l(k+1) = -\log\left(1 - O_l(\mathbf{p}(k))\right) p_l(k) \quad \forall\, l. \tag{12}$$

2) Normalize $\mathbf{p}(k+1)$:

$$\mathbf{p}(k+1) \leftarrow \frac{\mathbf{p}(k+1) \cdot \bar{P}}{\mathbf{1}^\top \mathbf{p}(k+1)} \;\text{ if }\; \mathcal{P} = \{\mathbf{p} \mid \mathbf{1}^\top \mathbf{p} \leq \bar{P}\}. \tag{13}$$

$$\mathbf{p}(k+1) \leftarrow \frac{\mathbf{p}(k+1)\cdot \bar{p}}{\max_j p_j(k+1)} \;\text{ if }\; \mathcal{P} = \{\mathbf{p} \mid p_l \leq \bar{p}\; \forall\, l\}. \tag{14}$$

Theorem 1: Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm 1 converges geometrically fast to the optimal solution of (7).

Proof: Let us write the left-hand side of the lth outage constraint in (9) as $\frac{f_l(\mathbf{p})}{p_l} \leq \alpha$, where

$$f_l(\mathbf{p}) = v_l\beta_l + \sum_j p_l \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right). \tag{15}$$

In the following, we show that $f_l(\mathbf{p})$ is a positive *concave self-mapping* on the standard cone $K = \mathbb{R}_+^L$. The definition of a concave self-mapping is given in [10] as follows.

Definition 1 (Concave Self-mapping [10]): A mapping $T : K \rightarrow K$ is concave if

$$T(a\mathbf{x}+(1-a)\mathbf{y}) \geq aT\mathbf{x}+(1-a)T\mathbf{y} \;\; \forall\, \mathbf{x}, \mathbf{y} \in K, \;\; a \in [0,1],$$

and monotone if $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $\mathbf{0} \leq T\mathbf{x} \leq T\mathbf{y}$.
Let $\|\cdot\|$ be a norm on $\mathbb{R}^L$ that is monotone, i.e., $\|\mathbf{x}\| \leq \|\mathbf{y}\|$. A concave self-mapping of K is monotone on K and continuous on the interior of K with respect to $\|\cdot\|$ [10].

We first show that $T = f_l(\mathbf{p})$ is a cone mapping with respect to the interior of K. Taking the derivative of $f_l(\mathbf{p})$ with respect to p_l, the jth entry of the first derivative $\triangledown f_l(\mathbf{p})$ is given by:

$$(\triangledown f_l(\mathbf{p}))_j = \begin{cases} \displaystyle\sum_k \left(\log\left(1 + \frac{\beta_l F_{lk} p_k}{p_l}\right) - \frac{\beta_l F_{lk} p_k}{p_l + \beta_l F_{lk} p_k}\right), & \text{if } j = l \\ \displaystyle\frac{\beta_l F_{lj} p_l}{p_l + \beta_l F_{lj} p_j}, & \text{if } j \neq l. \end{cases}$$

Since $z/(1+z) \leq \log(1+z)$ for $z \geq 0$, $(\triangledown f_l(\mathbf{p}))_l \geq 0$. Thus, $(\triangledown f_l(\mathbf{p}))_j \geq 0$ for all j, i.e., $f_l(\mathbf{p})$ increases monotonically in $\mathbf{p}$. Now, we state the following result [16].

Theorem 2 (Proposition 3.2 in [16]): Let $\mathcal{K}$ be the set of cone mappings with respect to the interior of the positive standard cone. Suppose $T : K \rightarrow K$ is differentiable and the following inequalities hold for the component mappings: $T_l : K \rightarrow \mathbb{R}_+$ for all l: $\sum_j \mid \frac{\partial T_l}{\partial p_j}(\mathbf{p}) \mid \leq T_l\mathbf{p}$ on K. Then $T \in \mathcal{K}$.

Now, we have

$$\begin{aligned} \textstyle\sum_j \mid \frac{\partial T_l}{\partial p_j}(\mathbf{p}) \mid &= \sum_j p_j (\triangledown f_l(\mathbf{p}))_j \\ &= \sum_k \left(p_l \log\left(1 + \frac{\beta_l F_{lk} p_k}{p_l}\right) - \frac{\beta_l F_{lk} p_k p_l}{p_l + \beta_l F_{lk} p_k}\right) \\ &\quad + \sum_{j \neq l} \frac{\beta_l F_{lj} p_l p_j}{p_l + \beta_l F_{lj} p_j} \;= \sum_k p_l \log\left(1 + \frac{\beta_l F_{lk} p_k}{p_l}\right) \leq f_l(\mathbf{p}) \end{aligned} \tag{16}$$

Hence, by Theorem 2, $f_l(\mathbf{p})$ is a strictly positive and monotone cone mapping on K.

We next show that $T = f_l(\mathbf{p})$ is a concave self-mapping. Taking the second derivative, we obtain the Hessian $\triangledown^2 f_l(\mathbf{p})$ with entries given by:

$$(\triangledown^2 f_l(\mathbf{p}))_{jk} = \begin{cases} -\dfrac{(\beta_l F_{lj})^2 p_j}{(p_l + \beta_l F_{lj} p_j)^2}, & \text{if } j = k, k \neq l \\ \dfrac{(\beta_l F_{lj})^2 p_j}{(p_l + \beta_l F_{lj} p_j)^2}, & \text{if } j \neq k, \text{ either } k \text{ or } j = l \\ -\displaystyle\sum_m \frac{(\beta_l F_{lm})^2 p_m^2 / p_l}{(p_l + \beta_l F_{lm} p_m)^2}, & \text{if } j = k = l \\ 0, & \text{otherwise.} \end{cases}$$

Now, the Hessian $\triangledown^2 f_l(\mathbf{p})$ is indeed negative definite: for all real vectors $\mathbf{z}$, we have

$$\mathbf{z}^\top \triangledown^2 f_l(\mathbf{p}) \mathbf{z} = -\frac{1}{p_l} \sum_k \frac{(\beta_l F_{lk})^2 (p_l z_k - p_k z_l)^2}{(p_l + \beta_l F_{lk} p_k)^2} < 0.$$

Another proof is to observe that $t\log(1 + x/t)$ is strictly concave in (x, t) for strictly positive t, as it is the perspective function of the strictly concave function $\log(1 + x)$ [15]. Hence, $f_l(\mathbf{p})$ is a sum of strictly concave perspective function, and therefore $f(\mathbf{p})$ is strictly concave in $\mathbf{p}$.

We note that any concave self-mapping of K is in $\mathcal{K}$ and it is monotone and continuous [16]. Indeed, $f_l(\mathbf{p})$ is monotone increasing in $\mathbf{p}$ as has been shown earlier.

We first state the following key theorem in [10].

Theorem 3 (Krause's theorem [10]): Let $\|\cdot\|$ be a monotone norm on $\mathbb{R}^L$. For a concave mapping $f : \mathbb{R}_+^L \rightarrow \mathbb{R}_+^L$ with $f(\mathbf{z}) > 0$ for $\mathbf{z} \geq \mathbf{0}$, the following statements hold. The conditional eigenvalue problem $f(\mathbf{z}) = \lambda\mathbf{z}$, $\lambda \in \mathbb{R}$, $\mathbf{z} \geq \mathbf{0}$, $\|\mathbf{z}\| = 1$ has a unique solution $(\lambda^*, \mathbf{z}^*)$, where $\lambda^* > 0$, $\mathbf{z}^* > \mathbf{0}$. Furthermore, $\lim_{k\rightarrow\infty} \tilde{f}(\mathbf{z}(k))$ converges geometrically fast to $\mathbf{z}^*$, where $\tilde{f}(\mathbf{z}) = f(\mathbf{z})/\|f(\mathbf{z})\|$.
The total and individual power constraints in (4) are the monotone norm $\|\mathbf{p}\|_1 = \bar{P}$ and $\|\mathbf{p}\|_\infty = \bar{p}$ respectively. By Theorem 3, the convergence of the iteration

$$\mathbf{p}(k+1) = f(\mathbf{p}(k))/\|f(\mathbf{p}(k))\|$$

to the *unique* fixed point $\mathbf{p} = f(\mathbf{p})/\|f(\mathbf{p})\|$ is geometrically fast, regardless of the initial point. ■

Remark 2: To compute $O_l(\mathbf{p}(k))$ in (12), the lth user measures separately the received interfering power $\{F_{lj}p_j(k)\}, j \neq l$. The normalization at Step 2 can be made distributed using gossip algorithms to compute either $\max_l p_l(k+1)$ or $\mathbf{1}^\top\mathbf{p}(k+1)$ [17].

In the following, we first derive useful bounds to $O^\star$ given in terms of the problem parameters of (7). Next, we solve (7) analytically in the interference-limited special case without any power constraint, and then extend the analysis to the general case with power constraints.

A. Worst Outage Probability Bounds

We now develop lower and upper bounds for the worst outage probability $O^\star$ using the certainty-equivalent margin

(CEM) problem, which is defined as the following maximization of the minimum weighted SINR problem:[4]

$$\begin{array}{ll} \text{maximize} & \min_{l} \dfrac{\mathsf{SINR}_l(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{1}^\top \mathbf{p} \le \bar{P},\ \mathbf{p} \ge \mathbf{0}. \end{array} \tag{17}$$

Now, the optimal value and solution of (17) can be obtained analytically [11]. We have the following bounds using the CEM analytical optimal value (also in terms of the constant problem parameters of (9)).

Corollary 1: If $\mathcal{P} = \{\mathbf{p} \mid p_l \le \bar{p}\ \forall\, l\}$, the worst outage probability $O^\star$ satisfies

$$\begin{aligned} \frac{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))}{1+\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))} &\le O^\star = 1 - e^{-\alpha^\star} \\ &\le 1 - e^{-\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))}, \end{aligned} \tag{18}$$

$$\text{where}\ \ i = \arg\max_{l} \rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_l^\top)) \tag{19}$$

and $\alpha^\star$ is the optimal value to (9).

Proof: Using the inequalities $1 + \sum_{l=1}^{L} z_l \le \prod_{l=1}^{L}(1+z_l) \le e^{\sum_{l=1}^{L} z_l}$ for nonnegative $\mathbf{z}$ (cf. [7]), a lower and upper bound on $\alpha^\star$ can be given by $1/(1+\mathrm{CEM}) \le \alpha^\star \le 1 - e^{-1/\mathrm{CEM}}$, where CEM is the optimal value of (17) and is given by $1/\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))$, where i is given by (19) [11]. The bounds on $O^\star = 1-e^{-\alpha^\star}$ can thus be obtained, hence proving Corollary 1. ∎

Remark 3: Note that the lower bound in Corollary 1 is not necessarily the tightest, but the bounds in Corollary 1 illustrate that the CEM problem, i.e., the spectral information of a concave self-mapping $T\mathbf{p} = \mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+\mathbf{v})\mathbf{p}$ (cf. [11]) can provide useful quick bounds to the worst outage probability. Corollary 1 reduces to a result in [7] in the interference-limited case. Results similar to Corollary 1 can also be obtained for the case when $\mathcal{P} = \{\mathbf{p} \mid \mathbf{1}^\top\mathbf{p} \le \bar{P}\}$, but is omitted here.

Example 1: Figure 1 plots the worst outage probability and the bounds for a system with 20 femtocell users using parameters in [1], where each user has a common SINR threshold β. Observe that the bounds are concave in β, and the worst outage probability is very close to its upper bound. In fact, our numerical observations indicate that the optimal power $\mathbf{p}^\star$ are also close in value to the optimal solution of (17), i.e., $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))$, where i is given by (19) (cf. [11]). Further, for small β, the bounds are very close, which suggests that in low-power femtocell networks, the CEM solution can give sufficiently good approximation to the worst outage probability.

B. Interference-limited Case

We now turn to solve (7) analytically for the interference-limited case, i.e., $\mathbf{v} = \mathbf{0}$. In this case, (15) is in addition a primitive positive homogeneous function of degree 1. Next,

[4] The CEM problem replaces the statistical variation in the desired signal and the interference of (3) by their mean values, and is a nonconvex problem that can also be solved by the nonlinear Perron-Frobenius theory [11].

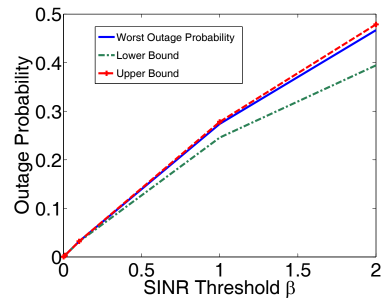

Fig. 1. Outage probability versus the SINR threshold β for a system with 20 femtocell users, where each user has a common SINR threshold β.

we define the nonnegative matrix $\mathbf{B}$ with the entries (that are functions of $\mathbf{p}$):

$$B_{lj} = \begin{cases} 0, & \text{if } l = j \\ \dfrac{p_l}{\beta_l p_j}\log\left(1 + \dfrac{\beta_l F_{lj} p_j}{p_l}\right), & \text{if } l \ne j. \end{cases} \tag{20}$$

Note that $\mathbf{B}(\mathbf{p})$ is irreducible whenever $\mathbf{F}$ is. Using (20), we can write the optimal value $\alpha^\star$ and optimal solution $\mathbf{p}^\star$ of (9) in the interference-limited special case as

$$\alpha^\star = \rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}^\star)) \tag{21}$$

$$\text{and } \mathbf{p}^\star = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}^\star))\ \ \text{(up to a scaling constant)} \tag{22}$$

respectively. Thus, $\mathbf{p}^\star$ is a fixed point of

$$\mathbf{p} = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}))}\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p})\mathbf{p}. \tag{23}$$

Further, it is interesting to note the following result of $\mathbf{p}^\star$.

Corollary 2: In the interference-limited case, the optimal power of (9), $\mathbf{p}^\star$, satisfies

$$\mathbf{p}^\star = \arg\max_{\mathbf{p}>\mathbf{0}} \rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p})). \tag{24}$$

Proof: Using (20), we can rewrite the outage constraints in (9) in matrix form as

$$\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p})\mathbf{p} \le \alpha\mathbf{p}. \tag{25}$$

Next, we need the following result from [13].

Theorem 4 (Theorem 1.6, [13] (Subinvariance Theorem)): Let $\mathbf{A}$ be an irreducible nonnegative matrix, s a positive number, and $\mathbf{z} \ge \mathbf{0}$, a vector satisfying $\mathbf{A}\mathbf{z} \le s\mathbf{z}$. Then, (i) $\mathbf{z} > \mathbf{0}$; (ii) $s \ge \rho(\mathbf{A})$. Moreover, $s = \rho(\mathbf{A})$ if and only if $\mathbf{A}\mathbf{z} = s\mathbf{z}$.

Applying Theorem 4 to (25) (let $\mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p})$, $\mathbf{z}$ be a feasible $\mathbf{p}$ and $s = \alpha$), we have $\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p})) \le \alpha$ for any feasible $\mathbf{p}$ and α. But $\alpha^\star = \rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}^\star))$. Hence, $\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p})) \le \rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}^\star))$. ∎

The following method was first proposed in [7] to compute the optimal solution for the interference-limited case without any power constraint:

$$\mathbf{p}(k+1) = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}(k)))}\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}(k))\mathbf{p}(k). \tag{26}$$

The authors in [7] observed that this iteration converges numerically to an acceptable accuracy with a fixed initial vector $\mathbf{p}(0) = \mathbf{x}(\text{diag}(\boldsymbol{\beta})\mathbf{F})$ (optimal solution of (17) without power constraints and noise power). The issues of convergence and existence of a fixed point were however left open in [7].

Our Algorithm 1 in fact reduces to an update similar to (26) when $\mathbf{v} = \mathbf{0}$, and computes a solution that is equal to that of (26) up to a scaling constant. The scaling factor in (26) tends to $\rho(\text{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}^\star))$ with increasing k. From Theorem 1, this means that (26) converges from *any initial point* to the fixed point in (23) geometrically fast, thus resolving an open problem of convergence in [7].

C. Duality by Lagrange and Perron-Frobenius

We now show that the optimal value $\alpha^\star$ and optimal solution $\mathbf{p}^\star$ of (9) can be derived analytically from the spectral information of a specially constructed rank-one perturbation of $\text{diag}(\boldsymbol{\beta})\mathbf{B}$, where $\mathbf{B}$ is given in (20). The following result is obtained based on the nonlinear Perron-Frobenius theory in [9] and the Friedland-Karlin inequality in [12].

Lemma 2: The optimal solution $(\mathbf{p}^\star, \alpha^\star)$ of (9) satisfies

$$\begin{aligned}
\log \alpha^\star &= \log \rho(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + \mathbf{v}\mathbf{c}_*^\top)) \\
&= \max_{\|\mathbf{c}\|_D = 1} \log \rho(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + \mathbf{v}\mathbf{c}^\top)) \\
&= \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \min_{\mathbf{p} \in \mathcal{P}} \sum_l \lambda_l \log \frac{\beta_l(\mathbf{B}(\mathbf{p})\mathbf{p} + \mathbf{v})_l}{p_l} \quad (27) \\
&= \min_{\mathbf{p} \in \mathcal{P}} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \sum_l \lambda_l \log \frac{\beta_l(\mathbf{B}(\mathbf{p})\mathbf{p} + \mathbf{v})_l}{p_l}, \quad (28)
\end{aligned}$$

where the optimal $\mathbf{p}$ in (27) and (28) are both given by $\mathbf{x}(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + \mathbf{v}\mathbf{c}_*^\top))$ (which is equal to $\mathbf{p}^\star$ up to a scaling constant), and the optimal $\boldsymbol{\lambda}$ in (27) and (28) are both given by the Hadamard product of $\mathbf{x}(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + \mathbf{b}\mathbf{c}_*^\top))$ and $\mathbf{y}(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + \mathbf{b}\mathbf{c}_*^\top))$.

Furthermore, $\mathbf{p} = \mathbf{x}(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + \mathbf{v}\mathbf{c}_*^\top))$ is the dual of $\mathbf{c}_*$ with respect to $\|\cdot\|_D$.[5]

Proof: Lemma 2 is proved by considering the Lagrange duality of (10) and then applying nonnegative matrix theory in [9], [12]. ■

Note that the optimal dual variable in (10) is equal to the optimal $\boldsymbol{\lambda}$ in (27) and (28). Using Lemma 2, we can now give analytically the optimal value and solution of (9) when $\mathcal{P} = \{\mathbf{p} \mid p_l \leq \bar{p} \, \forall \, l\}$.

Corollary 3: The optimal value and solution of (9) is given by

$$\alpha^\star = \rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)\right) \tag{29}$$

and

$$\mathbf{p}^\star = \mathbf{x}\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)\right), \tag{30}$$

$$\text{where } i = \arg\max_l \rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_l^\top)\right). \tag{31}$$

Furthermore, $p_i^\star = \bar{p}$ for the i (not necessarily unique) in (31).

[5]A pair $(\mathbf{x}, \mathbf{y})$ of vectors of $\mathbb{R}^L$ is said to be a dual pair with respect to $\|\cdot\|$ if $\|\mathbf{y}\|_D \|\mathbf{x}\| = \mathbf{y}^\top \mathbf{x} = 1$.

Remark 4: In general, the optimal index i in (31) differs from (19). Our simulations show that both are empirically the same when the CEM solution is close to $\mathbf{p}^\star$, especially so in the low-power regime (cf. Fig. 1). Unlike (31), (19) can be computed *a priori* from the problem parameters.

Table I summarizes the connection between the nonlinear Perron-Frobenius spectrum (of the respectively different concave self-mappings) and the optimal value and solution of the optimization problems under the CEM model and the Rayleigh-fading model subject to the different power constraints (individual and total power constraints).

IV. Total Power Minimization and Adaptive Outage Power Control

In this section, we first study the total power minimization problem subject to both outage specification and individual power constraints, and address its feasibility conditions using our results in previous sections. We then propose an adaptive algorithm to minimize the total power consumption and guarantee a min-max fairness in terms of worst outage probability.

The problem of minimizing the total power subject to given outage specification under Rayleigh fading and individual power constraints can be formulated as

$$\begin{array}{ll}
\text{minimize} & \mathbf{1}^\top \mathbf{p} \\
\text{subject to} & 1 - e^{\frac{-v_l \beta_l}{p_l}} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1} \leq \bar{O}_l \quad \forall \; l, \\
& \mathbf{p} \in \{\mathbf{p} \mid p_l \leq \bar{p} \;\; \forall \, l\}, \\
\text{variables:} & \mathbf{p},
\end{array} \tag{32}$$

where $0 < \bar{O}_l < 1$ is a given outage probability bound for the lth user. Depending on the given parameters $\bar{O}_l$ for all l, (32) may or may not be feasible. This is unlike the worst outage probability problem in (5), which is always feasible.

Next, using (15), (32) can be rewritten as

$$\begin{array}{ll}
\text{minimize} & \mathbf{1}^\top \mathbf{p} \\
\text{subject to} & \dfrac{f_l(\mathbf{p})}{p_l} \leq \alpha_l \quad \forall \; l, \\
& \mathbf{p} \in \{\mathbf{p} \mid p_l \leq \bar{p} \;\; \forall \, l\},
\end{array} \tag{33}$$

where $\alpha_l = -\log(1 - \bar{O}_l)$ for all l.

If (32) is feasible, it can be shown that all the L outage constraints in (32) are tight at optimality [8]. We deduce the feasibility condition of (32) from (33) in the following result.

Lemma 3: There is a unique and finite optimal $\mathbf{p}$ in (32) if and only if

$$\max_{\mathbf{p} \in \{\mathbf{p} \mid p_l \leq \bar{p} \, \forall \, l\}} \rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p})) < 1. \tag{34}$$

Furthermore, $\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p})) < 1$ if $\alpha^\star < \min_l \alpha_l$, i.e., $1 - e^{-\alpha^\star} < \bar{O}_l$ for all l.

Proof: To show the feasibility condition, we examine the condition under which there is a fixed point to

$$f_l(\mathbf{p}) = \beta_l((\mathbf{B}(\mathbf{p})\mathbf{p})_l + v_l) = \alpha_l p_l \;\; \text{for all} \;\; l,$$

which can be rewritten in matrix form as

$$(\mathbf{I} - \text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))\mathbf{p} = \text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{v}. \tag{35}$$

TABLE I
NONLINEAR PERRON-FROBENIUS CHARACTERIZATION OF THE CEM PROBLEM AND THE WORST OUTAGE PROBABILITY PROBLEM: THE SECOND AND THIRD ROW TABLUATE THE CEM CASE FOR INDIVIDUAL AND TOTAL POWER CONSTRAINTS RESPECTIVELY. THE FOURTH AND FIFTH ROW TABLUATE THE WORST OUTAGE PROBABILITY CASE FOR INDIVIDUAL AND TOTAL POWER CONSTRAINTS RESPECTIVELY.

Concave Self-mapping $(T\mathbf{p})_l$	*Perron eigenvalue* $\alpha^\star$	*Perron eigenvector* $\mathbf{p}^\star$	Remark
$(\text{diag}(\boldsymbol{\beta})(\mathbf{F}\mathbf{p}+(1/\bar{p})\mathbf{v}))_l$, $i=\arg\max_l \rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_l^\top)\right)$	$\rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)\right)$	$\mathbf{x}\left(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)\right)$	[11]
$(\text{diag}(\boldsymbol{\beta})(\mathbf{F}\mathbf{p}+(1/\bar{P})\mathbf{v}))_l$	$\rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{P})\mathbf{v}\mathbf{1}^\top)\right)$	$\mathbf{x}\left(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{P})\mathbf{v}\mathbf{1}^\top)\right)$	[18]
$v_l\beta_l+\sum_j p_l\log\left(1+\frac{\beta_l F_{lj}p_j}{p_l}\right)$ $i=\arg\max_l \rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star)+(1/\bar{p})\mathbf{v}\mathbf{e}_l^\top)\right)$	$\rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star)+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)\right)$	$\mathbf{x}\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star)+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)\right)$	Herein, Corollary 3
$v_l\beta_l+\sum_j p_l\log\left(1+\frac{\beta_l F_{lj}p_j}{p_l}\right)$	$\rho\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star)+(1/\bar{P})\mathbf{v}\mathbf{1}^\top)\right)$	$\mathbf{x}\left(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star)+(1/\bar{P})\mathbf{v}\mathbf{1}^\top)\right)$	Herein

We first state the following result from [13].

Theorem 5: A necessary and sufficient condition for a solution $\mathbf{z}\geq\mathbf{0},\mathbf{z}\neq\mathbf{0}$ to the equations $(\mathbf{I}-\mathbf{A})\mathbf{z}=\mathbf{c}$ to exist for any $\mathbf{c}\geq\mathbf{0},\mathbf{c}\neq\mathbf{0}$ is that $\rho(\mathbf{A})<1$. In this case there is only one solution $\mathbf{z}$, which is strictly positive and given by $\mathbf{z}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{c}$.

Since $\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}$ is an irreducible nonnegative matrix, it follows from Theorem 5 (letting $\mathbf{A}=\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}$, $\mathbf{c}=\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{v}$) that $\mathbf{p}$ in (35) is unique and strictly positive if and only if $\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))<1$ for all $\mathbf{p}\in\{\mathbf{p}\mid p_l\leq\bar{p}\,\forall\,l\}$. This is equivalent to stating

$$\max_{\mathbf{p}\in\{\mathbf{p}|p_l\leq\bar{p}\,\forall\,l\}}\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))<1,$$

thus proving the first part.

To show the second part, we note that $\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))\leq(1/\min_l\alpha_l)\rho(\text{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}))\leq(1/\min_l\alpha_l)\alpha^\star$. Thus, a sufficient condition that $(1/\min_l\alpha_l)\alpha^\star<1$ implies that $\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))<1$. ■

A. Feasibility Bounds

From (34) in Lemma 3, we see that verifying the feasibility of (32) requires solving a Perron-Frobenius eigenvalue maximization problem (a nonconvex problem). We next provide useful (tight) bounds on this nonconvex optimal value $\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))$ in (34) that exploit the optimal value and solution of the worst outage probability problem in Section III.

Theorem 6: Let $\alpha^\star$ and $\mathbf{p}^\star$ be given in (21) and (22), respectively. Then, we have

$$\begin{aligned}&\prod_l(\alpha_l)^{-x_l(\mathbf{B}(\mathbf{p}^\star))y_l(\mathbf{B}(\mathbf{p}^\star))}\alpha^\star\leq\\&\max_{\mathbf{p}\in\{\mathbf{p}|p_l\leq\bar{p}\,\forall\,l\}}\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))\leq\max_l(\alpha^\star/\alpha_l).\end{aligned}\tag{36}$$

Further, equality is achieved in the lower and upper bounds when α_l's are equal for all l.

Proof: By applying the Friedland-Karlin inequalities in [12], the function $\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p}))$ can be bounded by

$$\begin{aligned}&\prod_l(\alpha_l)^{-x_l(\mathbf{B})y_l(\mathbf{B})}\rho(\text{diag}(\boldsymbol{\beta})\mathbf{B})\leq\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B})\\&\leq\max_l(1/\alpha_l)\rho(\text{diag}(\boldsymbol{\beta})\mathbf{B})\end{aligned}\tag{37}$$

for any feasible $\mathbf{p}\in\{\mathbf{p}\mid p_l\leq\bar{p}\,\forall\,l\}$. Now, from the lower bound in (37), we have

$$\begin{aligned}&\prod_l(\alpha_l)^{-x_l(\mathbf{B}(\mathbf{p}^\star))y_l(\mathbf{B}(\mathbf{p}^\star))}\rho(\text{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}^\star))\\&\leq\max_{\mathbf{p}\in\{\mathbf{p}|p_l\leq\bar{p}\,\forall\,l\}}\left\{\prod_l(\alpha_l)^{-x_l(\mathbf{B}(\mathbf{p}))y_l(\mathbf{B}(\mathbf{p}))}\rho(\text{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}))\right\}\\&\leq\max_{\mathbf{p}\in\{\mathbf{p}|p_l\leq\bar{p}\,\forall\,l\}}\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p})),\end{aligned}\tag{38}$$

where $\mathbf{p}^\star\in\{\mathbf{p}\mid p_l\leq\bar{p}\,\forall\,l\}$ is given by (22). Using $\alpha^\star$ given in (21), we thus establish (36). The condition under which equalities in (37) are achieved follows from the application of the Friedland-Karlin inequalities in [12]. ■

These bounds can be easily computed in the two-user case.[6]

Example 2: In the two-user case, we have

$$\frac{\alpha^\star}{\sqrt{\alpha_1\alpha_2}}\leq\max_{\mathbf{p}\in\{\mathbf{p}|p_l\leq\bar{p}\,\forall\,l\}}\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B})\leq\max\left\{\frac{\alpha^\star}{\alpha_1},\frac{\alpha^\star}{\alpha_2}\right\}.$$

Combining Theorem 6 with Corollary 1, simplified bounds in terms of the CEM solution (and more directly in terms of the problem parameters) can be obtained:

$$\begin{aligned}&\prod_l(\alpha_l)^{-x_l(\mathbf{B}(\mathbf{p}^\star))y_l(\mathbf{B}(\mathbf{p}^\star))}\log(1+\rho(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)))\\&\leq\max_{\mathbf{p}\in\{\mathbf{p}|p_l\leq\bar{p}\,\forall\,l\}}\rho(\text{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B})\\&\leq\max_l(1/\alpha_l)\rho(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top)),\end{aligned}$$

where i is given by (19) and we have made use of the fact that $\max_l\rho(\text{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star)+(1/\bar{p})\mathbf{v}\mathbf{e}_l^\top))\leq\max_l\rho(\text{diag}(\boldsymbol{\beta})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_l^\top))$ in the last inequality.

[6]The Schur product of the Perron and left eigenvectors of a zero-diagonal 2×2 positive matrix equals $[1/2,1/2]$, simplifying the computation in (36).

We now state the following algorithm proposed in [8].

Algorithm 2 (Total Power Minimization):

$$p_l(k+1) = \min\{-\log(1 - O_l(\mathbf{p}(k)))\, p_l(k)/\alpha_l,\ \bar{p}\}\ \forall\, l. \tag{39}$$

We now establish the necessary and sufficient condition under which Algorithm 2 converges. This condition is also necessary and sufficient for (32) to be feasible.

Corollary 4: Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm 2 converges geometrically fast to the optimal solution of (32) if and only if $\rho(\mathrm{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha})\mathbf{B}(\mathbf{p})) < 1$ for all $\mathbf{p} \in \{\mathbf{p} \mid p_l \le \bar{p}\, \forall\, l\}$.

Proof: The necessary and sufficient condition under which (32) is feasible is given in Lemma 3. If (32) is feasible, the convergence proof for Algorithm 2 can be found in [8]. Hence, Corollary 4 is proved. ■

B. Adaptive Outage-based Power Control

In this section, we propose an adaptive outage-based power control algorithm for total energy minimization.

Algorithm 3 (Adaptve Outage-based Power Control (AOPC)):

1) Update the auxiliary variable $\mathbf{z}(k+1)$:

$$z_l(k+1) = -\log(1 - O_l(\mathbf{z}(k)))\, z_l(k) \quad \forall\, l. \tag{40}$$

2) Normalize $\mathbf{z}(k+1)$:

$$\mathbf{z}(k+1) \leftarrow \mathbf{z}(k+1) \cdot \bar{p} / \max_j z_j(k+1). \tag{41}$$

3) Update the transmit power $\mathbf{p}(k+1)$:

$$p_l(k+1) = \min\left\{\frac{-\log(1 - O_l(\mathbf{p}(k)))\, p_l(k)}{\max\{\alpha_l, -\log(1 - O_l(\mathbf{z}(k)))\}},\ \bar{p}\right\} \forall\, l. \tag{42}$$

Corollary 5: Starting from any initial point $\mathbf{z}(0)$ and $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm 3 converges geometrically fast to the optimal solution of (33), where the righthand side of the outage constraints in (33) are replaced by $\max\{\alpha_l,\ \rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))\}$, where $\mathbf{p}^\star$ and i are given by (30) and (31) respectively, for all l.

Proof: Theorem 1 proves the convergence of $\mathbf{z}(k)$ in Step 1 and 2 of Algorithm 3, and also $\lim_{k\to\infty} -\log(1 - O_l(\mathbf{z}(k))) \to \rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))$. From Corollary 4, $\mathbf{p}(k)$ converges to a point $\mathbf{p}'$ that satisfies $-\log(1 - O_l(\mathbf{p}')) = \boldsymbol{\alpha}' = \max\{\boldsymbol{\alpha},\ \rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))\}$, which always satisfies $\rho(\mathrm{diag}(\boldsymbol{\beta}/\boldsymbol{\alpha}')\mathbf{B}(\mathbf{p})) < 1$. This proves Corollary 5. ■

Remark 5: If $\alpha_l < \rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))\}$ for all l, then $\lim_{k\to\infty} \mathbf{z}(k) = \lim_{k\to\infty} \mathbf{p}(k) = \mathbf{p}^\star$ in Algorithm 3.

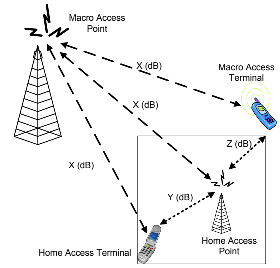

Fig. 2. A macro-femtocell model. The parameter X dB, Y dB and Z dB denote pass gain between MAP and HAP, between HAP and HAT and between HAP and MAT respectively (cf. Table 2 of [1] for values of X, Y, Z).

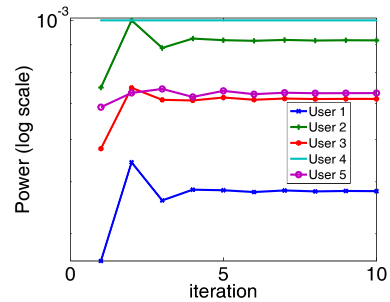

Fig. 3. Experiment 1. Convergence in power for five femtocell users using Algorithm 1.

V. Numerical Evaluation

In this section, we evaluate the performance of Algorithms 1, 2 and AOPC. Fig. 2 illustrates the basic macro-femtocell interference network model used in [1] for our simulations. A femtocell user is a link between the home access terminal and access point (box of Fig. 2), and interference comes from the macrocell base station/access terminals and other femtocell users. We consider a single macrocell base station with 50 macro access terminals and vary the number of femtocell users. We use the dense-urban propagation parameters in Table 2 of [1].

A. Expt. 1 (Convergence of Algorithm 1)

Fig. 3 shows the convergence of Algorithm 1 for five femtocell users. Simulations show that convergence happens in less than ten iterations even for thousands of users and a large power range, e.g., 125mW to 2W (maximum output of UMTS/3G Power Class 4 to Class 1 mobile phone respectively). From Theorem 3, Algorithm 1 can be viewed as a *nonlinear* power method in linear algebra. It is well known that the convergence rate of the power method is determined by the ratio of the second dominant eigenvalue to the Perron-Frobenius eigenvalue [13]. The method converges slowly if this ratio is close to one. Now, this ratio for $\mathbf{B}(\mathbf{p}^\star)$ (cf. fourth and fifth row of Table I) determines only the *local convergence rate* of Algorithm 1 near the fixed point $\mathbf{p}^\star$. Since the CEM solution is numerically observed to give good approximation to $\mathbf{p}^\star$ (especially in the regime of low-power and small $\boldsymbol{\beta}$), this ratio is conceivably close to those computed using the CEM

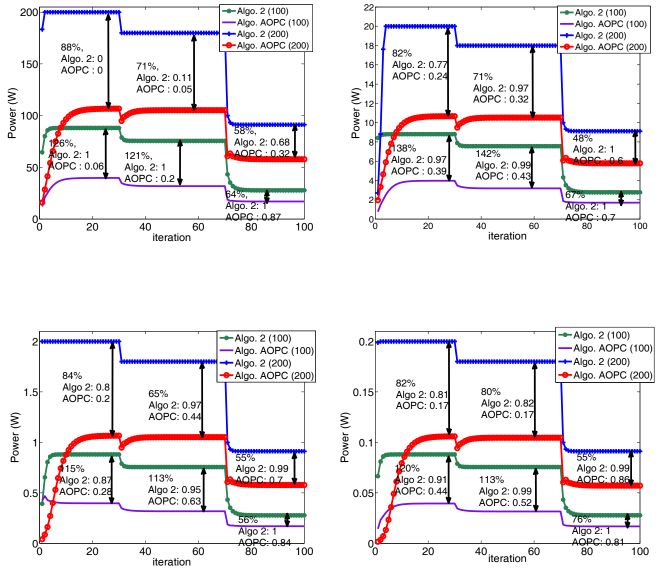

Fig. 4. Experiment 2. We plot the evolution of the total power consumption for two networks with 100 and 200 femtocell users as we vary $\bar{p}$. A time slot on the x-axis of each graph refers to a power update iteration. Ten and fifty percent of the users are removed at time slot 30 and 70 respectively. The top left graph, top right graph, bottom left graph and bottom right graph show the case for $\bar{p} = 1W$, $\bar{p} = 0.1W$, $\bar{p} = 10$mW and $\bar{p} = 1$mW respectively. Power savings between Algorithms 2 and AOPC, and the fraction of the number of users meeting the given threshold α are in text form.

solution (cf. second and third row of Table I), which on the other hand determines a *global convergence rate* for the CEM case. We empirically determine this ratio to be in the range of $0.2 - 0.4$ using the parameters in [1]. This thus determines numerically the overall convergence rate of Algorithm 1.

B. Expt. 2 (Comparison between Algorithms 2 and AOPC)

We next provide numerical examples to compare the total power consumption using Algorithms 2 and AOPC. Fig. 4 shows the total power evolution in a network with initially 100 and 200 femtocell users. Then, ten and fifty percent of the users are removed at time slot 30 and 70 respectively. On each graph, the difference in total power and the fraction of users that satisfy their outage probability threshold α are recorded. As illustrated, Algorithm 2 can produce power savings of 50% or more in all cases at the expense of a smaller number of users meeting α. On the other hand, problem infeasibility causes users who run Algorithm 2 and unable to achieve α to transmit at $\bar{p}$. By enforcing a worst outage probability fairness across all users, Algorithm AOPC computes power that are typically smaller than $\bar{p}$, thus leading to a smaller total power.

VI. Conclusion

We solved analytically the worst outage probability problem having either a total or individual power constraints in a multiuser Rayleigh-faded network. We then proposed a geometrically fast convergent algorithm to solve it optimally in a distributed manner. As a by-product, we solved an open problem of convergence for a previously proposed algorithm in the interference-limited case. We also established a tight relationship between the worst outage probability problem and its certainty-equivalent margin counterpart, and utilized the connection to find useful bounds and convergence rate. We then addressed a total power minimization problem with outage specification constraints and its feasibility condition. We proposed a dynamic algorithm that adapted its outage probability specification to minimize the total power in a heterogeneous network. This permitted a graceful handling of outage infeasibility in a distributed manner, and guaranteed a min-max fairness in terms of the worst outage probability.

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References

[1] M. Yavuz, F. Meshkati, S. Nanda, A. Pokhariyal, N. Johnson, B. Raghothaman, and A. Richardson. Interference management and performance analysis of UMTS-HSPA+ femtocells. *IEEE Communications Magazine*, 47(9):102–109, 2009.

[2] R. Madan, J. Borran, A. Sampath, N. Bhushan, A. Khandekar, and T. Ji. Cell association and interference coordination in heterogeneous LTE-A cellular networks. *IEEE Journal on Selected Areas in Communications*, 28(9):1479–1489, 2010.

[3] D. N. C. Tse and P. Viswanath. *Fundamentals of Wireless Communication*. Cambridge University Press, 1st edition, 2005.

[4] S. Jagannathan, M. Zawodniok, and Q. Shang. Distributed power control of cellular networks in the presence of Rayleigh fading channel. *Proc. of IEEE Infocom*, 2004.

[5] V. Chandrasekhar, J. G. Andrews, T. Muharemovic, Z. Shen, and A. Gatherer. Power control in two-tier femtocell networks. *IEEE Trans. on Wireless Communications*, 8(8):4316–4328, 2009.

[6] S. V. Hanly, L. H. Andrew, and T. Thanabalasingham. Dynamic allocation of subcarriers and transmit powers in an OFDMA cellular network. *IEEE Trans. on Information Theory*, 55(12):5445–5462, 2009.

[7] S. Kandukuri and S. Boyd. Optimal power control in interference-limited fading wireless channels with outage-probability specifications. *IEEE Trans. on Wireless Communications*, 1(1):46–55, Jan 2002.

[8] J. Papandriopoulos, J. Evans, and S. Dey. Optimal power control for Rayleigh-faded multiuser systems with outage constraints. *IEEE Trans. on Wireless Communications*, 4(6):2705–2715, November 2005.

[9] V. D. Blondel, L. Ninove, and P. Van Dooren. An affine eigenvalue problem on the nonnegative orthant. *Linear Algebra and its Applications*, 404:69–84, 2005.

[10] U. Krause. Concave Perron-Frobenius theory and applications. *Nonlinear analysis*, 47(2001):1457–1466, 2001.

[11] C. W. Tan, M. Chiang, and R. Srikant. Fast algorithms and performance bounds for sum rate maximization in wireless networks. *Proc. of IEEE Infocom*, 2009.

[12] S. Friedland and S. Karlin. Some inequalities for the spectral radius of non-negative matrices and applications. *Duke Mathematical Journal*, 42(3):459–490, 1975.

[13] E. Seneta. *Non-negative Matrices and Markov Chains*. Springer, 2nd edition, 2006.

[14] Y. D. Yao and A. Sheikh. Investigations into cochannel interference in microcellular mobile radio systems. *IEEE Trans. on Vehicular Technology*, 41(2):114–123, May 1992.

[15] S. Boyd and L. Vanderberghe. *Convex Optimization*. Cambridge University Press, 2004.

[16] U. Krause. A local$-$global stability principle for discrete systems and difference equations. *Proc. of 6th International Conference on Difference Equations 2001*, CRC Press:167–180, 2004.

[17] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Trans. on Information Theory*, 52(6):2508–2530, 2006.

[18] C. W. Tan, M. Chiang, and R. Srikant. Maximizing sum rate and minimizing MSE on multiuser downlink: Optimality, fast algorithms and equivalence via max-min SIR. *Proc. of IEEE ISIT*, 2009.