5. Three Dimensional Transformations

Methods for geometric transformations and object modelling in 3D are extended from 2D methods by including the considerations for the z coordinate.

Basic geometric transformations are: Translation, Rotation, Scaling

5.1 Basic Transformations

Translation

We translate a 3D point by adding translation distances, $t_x$, $t_y$, and $t_z$, to the original coordinate position $(x,y,z)$:

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$
Alternatively, translation can also be specified by the transformation matrix in
the following formula:

\[
\begin{bmatrix}
x'
y'
z'
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & t_x \\
0 & 1 & 0 & t_y \\
0 & 0 & 1 & t_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Exercise: translate a triangle with vertices at original coordinates (10,25,5),
(5,10,5), (20,10,10) by \( t_x=15, t_y=5, t_z=5 \). For verification, roughly plot
the x and y values of the original and resultant triangles, and imagine the
locations of z values.
Scaling With Respect to the Origin

We scale a 3D object with respect to the origin by setting the scaling factors $s_x$, $s_y$ and $s_z$, which are multiplied to the original vertex coordinate positions $(x,y,z)$:

$$x' = x \times s_x, \quad y' = y \times s_y, \quad z' = z \times s_z$$

Alternatively, this scaling can also be specified by the transformation matrix in the following formula:

$$\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} = \begin{bmatrix}
s_x & 0 & 0 & 0 \\
0 & s_y & 0 & 0 \\
0 & 0 & s_z & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}$$
Exercise: Scale a triangle with vertices at original coordinates (10,25,5), (5,10,5), (20,10,10) by $s_x=1.5$, $s_y=2$, and $s_z=0.5$ with respect to the origin. For verification, roughly plot the x and y values of the original and resultant triangles, and imagine the locations of z values.

Scaling with respect to a Selected Fixed Position

Exercise: What are the steps to perform scaling with respect to a selected fixed position? Check your answer with the text book.

Exercise: Scale a triangle with vertices at original coordinates (10,25,5), (5,10,5), (20,10,10) by $s_x=1.5$, $s_y=2$, and $s_z=0.5$ with respect to the centre of the triangle. For verification, roughly plot the x and y values of the original and resultant triangles, and imagine the locations of z values.
Coordinate-Axes Rotations

A 3D rotation can be specified around any line in space. The easiest rotation axes to handle are the coordinate axes.
Z-axis rotation: $x' = x \cos \theta - y \sin \theta$, 
$y' = x \sin \theta + y \cos \theta$, and 
$z' = z$

or:

$$\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  1
\end{bmatrix} = 
\begin{bmatrix}
  \cos \theta & -\sin \theta & 0 & 0 \\
  \sin \theta & \cos \theta & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}$$

X-axis rotation: $y' = y \cos \theta - z \sin \theta$, 
$z' = y \sin \theta + z \cos \theta$, and 
$x' = x$

or:
\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 & y' \\
0 & \sin \theta & \cos \theta & 0 & z' \\
1 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

Y-axis rotation:

\[
z' = z \cos \theta - x \sin \theta,
\]

\[
x' = z \sin \theta + x \cos \theta, \text{ and}
\]

\[
y' = y
\]

or:

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
1
\end{bmatrix} = \begin{bmatrix}
\cos \theta & 0 & \sin \theta & 0 \\
-\sin \theta & 1 & \cos \theta & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]
3D Rotations About an Axis Which is Parallel to an Axis

(a) Original Position of Object

(b) Translate Rotation Axis onto x Axis

(c) Rotate Object Through Angle $\theta$

(d) Translate Rotation Axis to Original Position
Step 1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.

Step 2. Perform the specified rotation about that axis.

Step 3. Translate the object so that the rotation axis is moved back to its original position.
General 3D Rotations

Initial Position

Step 1
Translate $P_1$ to the Origin

Step 2
Rotate $P_2'$ onto the $z$ Axis

Step 3
Rotate the Object Around the $z$ Axis

Step 4
Rotate the Axis to the Original Orientation

Step 5
Translate the Rotation Axis to the Original Position
Step 1. Translate the object so that the rotation axis passes through the coordinate origin.

Step 2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.

Step 3. Perform the specified rotation about that coordinate axis.

Step 4. Rotate the object so that the rotation axis is brought back to its original orientation.

Step 5. Translate the object so that the rotation axis is brought back to its original position.