Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Theoretical Computer Science 337 (2005) 390–398
Theoretical Computer Science
www.elsevier.com/locate/tcs

Note

Approximation of Walrasian equilibrium in single-minded auctions☆

Li-Sha Huang*, Minming Li, Bo Zhang

State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing, China

Received 23 August 2004; received in revised form 21 February 2005

Communicated by D.-Z. Du

Abstract

We consider a social optimization model of pricing scheme in single-minded auctions, in cases where Walrasian equilibrium does not exist. We are interested in the maximization of the ratio, R, of happy bidders over all agents, in a feasible allocation-pricing scheme. We show NP-hardness of the optimization problem, establish lower and upper bounds of R, as well as develop greedy algorithms to approximate the optimal value of R.

Keywords: Combinatorial auction; Walrasian equilibrium

1. Introduction

The algorithms of computing equilibria in an economy model have been adopted as one of the most significant research considerations in computational economics [6]. Among various equilibrium models, Walrasian equilibrium is a traditional definition of equilibrium with wonderful characteristics. In this work, we mainly consider the algorithmic issues of

☆ This work was supported by the National Natural Science Foundation of China (60135010, 60321002) and the Chinese National Key Foundation Research and Development Plan (2004CB318108).

* Corresponding author. Tel.: +86 1062775640.
E-mail addresses: hs02@mails.tsinghua.edu.cn (L.-S. Huang), liminming98@mails.tsinghua.edu.cn (M. Li), dcszb@mail.tsinghua.edu.cn (B. Zhang).

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2005.03.008

approximating Walrasian equilibrium in the combinatorial auction, which is another important research area that has attracted much attention [12] recently. In a combinatorial auction, the auctioneer sells heterogeneous types of indivisible commodities to some bidders. Different from traditional auctions, the bidders in a combinatorial auction bid on combinations of commodities. In general, each bidder can use a *value function* to represent his/her bids on bundles of commodities, which maps from the collection of all subsets of commodities to $\mathbb{R}^+$. In this paper, we concentrate on a subclass of combinatorial auctions, *single-minded auctions*. Bidders in this kind of auction are all *single-minded bidders*: they only bid on a certain bundle of auctioned items. Hence their value functions are greatly simplified. Although with a very simple form, this kind of auction is so powerful that every combinatorial auction can be converted to a single-minded auction by introducing virtual bidders and virtual commodities [10].

In a combinatorial auction, a Walrasian equilibrium can be viewed as an allocation-pricing scheme satisfying the following conditions:

- Each commodity is assigned a price, and the price of a bundle is the summation of the prices of all commodities in that bundle.
- Any unallocated commodity is assigned a price of value zero.
- Under these prices, the specific allocation guarantees that every bidder cannot gain more *utility* by any other allocation.

Kelso and Crawford [9] first proved the existence of Walrasian equilibrium under a certain condition. Later, Bikhchandani and Mamer [2] established a necessary and sufficient condition for the existence of Walrasian equilibrium by the duality theorem in linear programming. Recently, Chen et al. [4] studied the algorithmic issues of Walrasian equilibrium in single-minded auctions and proved that deciding the existence of Walrasian equilibrium in single-minded auctions is NP-hard.

Naturally, people will resort to approximating equilibrium in some sense. Postlewaite and Schmeidler [11] considered one of the possible approximation directions. They defined ε-Walrasian equilibrium which means every bidder is almost satisfied within a ratio $(1-\varepsilon)$ instead of the absolute satisfaction in normal Walrasian equilibriums.

In this paper, we consider the approximation in another direction inspired by the concept in the work of Demange [5] and Hansen and Thisse [8], which characterized the *majority equilibrium* in voting process. Informally speaking, in the decision making problem of public affairs, people would like to make decisions by voting with majority rule. In such a voting process, a majority equilibrium is a solution with the property that no other solutions can please more than half of the voters in comparison to it. Recently, Chen et al. [3] developed a fast algorithm to achieve the equilibrium for the cases with nice structures and proved the NP-hardness for the general case.

Similarly, regarding the auction as a special kind of voting, we aim to maximize the number of satisfied bidders instead of making every bidder approximately satisfied. In this way, we obtain a ratio R which is defined to be the maximal fraction of satisfied bidders in a single-minded auction. We observe that if no restrictions are placed on single-minded auctions, the ratio R cannot be satisfactorily high when the number of commodities m increases. Even if we require the number of commodities and the number of bidders to be approximately equal, we still cannot guarantee a satisfactory lower bound of R. We try to estimate the lower bound of R and get some positive results for small m.

The rest of the paper is organized as follows: Section 2 provides some necessary preliminaries and the formal definition of our problem. In Section 3, we give an NP-hardness proof for the optimization problem of maximizing R. We study the upper and lower bounds of R and provide a greedy algorithm to approximate the optimal value of R.

2. Model and definitions

A combinatorial auction is a triple (Ω, I, V) which consists of the following issues:

- *Commodities*: The auctioneer sells m indivisible commodities in the market. Let $\Omega = \{\omega_1, \ldots, \omega_m\}$ denote the set of commodities.
- *Agents*: There are n agents in the market acting as bidders, denoted by $I = \{1, 2, \ldots, n\}$.
- *Value functions*: Each bidder $i \in I$ has a value function $v_i : 2^\Omega \to \mathbb{R}^+$ to submit his bids on bundles of commodities. Denote all bidders' value functions by $V = \{v_1, v_2, \ldots, v_n\}$.

In this paper, we concentrate on *single-minded auctions*. In a single-minded auction, each bidder is only interested in his *basic bundle*. More precisely, for a single-minded bidder i, there exists a subset $\Omega_i \subseteq \Omega$ and a positive number w_i such that his value function v_i is defined by

$$v_i(E) = \begin{cases} w_i & \text{if } E = \Omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

If we denote a subset of Ω by a 0–1 vector in $\mathbb{R}^m$, then it is natural to represent a single-minded bidder's value function by a pair (a, w) where $a \in \{0, 1\}^m$ is the bidder's basic bundle and $w \in \mathbb{R}$ is his bid on his basic bundle. From now on, w will be called the bidder's *budget*. With these notations, a single-minded auction can be represented by a pair (A, w), where $A = (a^i_j)$ is a $m \times n$ matrix whose ith column a^i is the basic bundle of bidder i and $w = (w_i) \in \mathbb{R}^n$ is the bidders' budget vector.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of two vectors. Let e denote the vector of all 1's whose dimension is clear within the context.

An *allocation* $x = (x_1, x_2, \ldots, x_n)^T$ is an n-dimensional 0–1 vector whose ith entry is 1 if and only if bidder i wins his basic bundle. The *social welfare* of this allocation is defined by $\langle w, x \rangle$. An allocation x is *feasible* if $Ax \leqslant e$ which ensures every commodity is sold to at most one bidder.

A *price vector* p is a nonnegative vector in $\mathbb{R}^m$ whose jth entry is the price of good ω_j. Define the *utility* of bidder i under an allocation x and price p by

$$u_i(x, p) = \begin{cases} w_i - \langle a^i, p \rangle & \text{if } x_i = 1, \\ 0 & \text{if } x_i = 0. \end{cases}$$

A bidder is *satisfied* under an allocation x and price p if he cannot gain more utility in any other allocation $\hat{x}$. Obviously, bidder i is a satisfied winner (loser) if and only if $w_i \geqslant \langle a^i, p \rangle$ ($w_i \leqslant \langle a^i, p \rangle$).

Definition 1 (*Gul and Stacchetti [7]*). For a single-minded combinatorial auction (A, w), a Walrasian equilibrium is a tuple (x, p), where x is a feasible allocation and p is a price vector, satisfying that every bidder is satisfied and $\langle p, Ax \rangle = \langle p, e \rangle$.

The condition $\langle p, Ax\rangle = \langle p, e\rangle$ is referred as *market clearing price* in the literature.

Example 1 (*An example of Walrasian equilibrium*). There are three bidders $\{1, 2, 3\}$ and three commodities $\{\omega_1, \omega_2, \omega_3\}$ in the auction. Their value functions are $\left\{(1, 1, 0)^{\mathrm{T}}, 6\right\}$, $\left\{(1, 0, 1)^{\mathrm{T}}, 2\right\}$ and $\left\{(0, 1, 1)^{\mathrm{T}}, 3\right\}$, respectively. Then setting the allocation vector as $x = (1, 0, 0)^{\mathrm{T}}$ and price vector as $(2, 3, 0)^{\mathrm{T}}$ will satisfy all of the three bidders. However, if the second bidder's budget is raised to 4, there is no Walrasian equilibrium in such an auction.

In general, Walrasian equilibrium may not exist [2,4]. With this undesirable fact, Postlewaite and Schmeidler [11] defined ε-Walrasian equilibrium which means every bidder is satisfied in $(1 - \varepsilon)$ sense instead of the absolute satisfaction in Walrasian equilibrium. In this paper, we try to relax the constraints in the definition of equilibriums in another way:

Definition 2. For a single-minded auction (A, w), a feasible allocation-pricing scheme is a tuple (x, p), where x is a feasible allocation and p is a price vector, satisfying that every **winner** is satisfied and $\langle p, Ax\rangle = \langle p, e\rangle$.

Such a scheme always exists in a single-minded auction since any feasible allocation with prices of value zero can make all winners satisfied.

Given a feasible allocation-pricing scheme (x, p) in a single-minded auction (A, w), denote by $s(x, p)$ the number of satisfied bidders and $r(x, p) = s(x, p)/n$.

Definition 3. For a single-minded auction (A, w), a semi-Walrasian equilibrium is a feasible allocation-pricing scheme (x^*, p^*) which maximizes the function $s(x, p)$. Define

$$S(A, w) = s(x^*, p^*), \quad R(A, w) = r(x^*, p^*).$$

It is well known that Walrasian equilibrium leads to a maximum of social welfare [4]. However, the following example shows that it is not true for semi-Walrasian equilibrium:

Example 2. There are three commodities $\{\omega_1, \omega_2, \omega_3\}$ in the market. Bidder 1's value function is $\left\{(1, 1, 0)^{\mathrm{T}}, 6\right\}$. Bidders 2 and 3 share a value function $\left\{(1, 0, 1)^{\mathrm{T}}, 4\right\}$. Bidders 4 and 5 share a value function $\left\{(0, 1, 1)^{\mathrm{T}}, 4\right\}$. Allocating $\{\omega_1, \omega_2\}$ to bidder 1 will maximize the social welfare. But it can satisfy at most three bidders, whatever the prices are. On the other hand, allocating $\{\omega_1, \omega_3\}$ to bidder 2 and setting $p = (0, 0, 4)^{\mathrm{T}}$ will satisfy four bidders. It is a semi-Walrasian equilibrium, but the social welfare is not maximal.

3. NP-hardness of the problem

Since Walrasian equilibrium may not exist in general, it becomes an interesting problem to investigate when the equilibrium exists. Chen et al. [4] proved that the decision problem about the existence of an equilibrium is NP-hard:

Lemma 1 (*Chen et al. [4]*). *Given any single-minded auction, it is* NP-*hard to determine whether there exists a Walrasian equilibrium*.

We try to find a generalization of their result from Walrasian equilibrium to semi-Walrasian equilibrium. Obviously, $R(A, w) = 1$ if and only if Walrasian equilibrium exists. Hence as a corollary of Lemma 1, the optimization version of maximizing $r(x, p)$ is NP-hard. The following theorem shows that its decision version is also NP-hard for any small constant c:

Theorem 1. *For any constant* $0 < c \leqslant 1$, *it is* NP-*hard to determine whether* $R(A, w) > c$.

Before the proof of this theorem, we need a lemma:

Lemma 2. *For any positive integer n, there exists an auction* (A_n, ω_n) *with n bidders and* $n(n-1)/2$ *commodities such that* $S(A_n, w_n) = 2$.

Proof. Let $I = \{1, 2, \ldots, n\}$ be the set of bidders and the budget of each bidder is 1. For any two bidders i and j ($i < j$), introduce a commodity ω_{ij}. Hence the set of commodities is $\Omega = \left\{\omega_{ij} | 1 \leqslant i < j \leqslant n\right\}$. The basic bundle of bidder i is $\Omega_i = \left\{\omega_{ij} | i < j \leqslant n\right\} \cup \left\{\omega_{ji} | 1 \leqslant j < i\right\}$. Since every pair of bidders' basic bundles conflict, there is at most one winner in the auction. By the constraints of market clearing price and the construction of this auction, there is at most one satisfied loser. Hence $S(A_n, \omega_n) = 2$. □

Proof of Theorem 1. When $c = 1$, it is equivalent to Lemma 1.

When $0 < c < 1$, we carry out the proof by a reduction from the decision problem of Walrasian equilibrium.

For any single-minded auction (A, w) in which there are n bidders competing for m commodities, we construct another auction (A', w') by introducing new bidders and commodities.

These new bidders and commodities are constructed as in the proof of Lemma 2: there are $\tilde{n}$ bidders competing for $\tilde{m} = \tilde{n}(\tilde{n} - 1)/2$ commodities and only two bidders can be satisfied. We can choose an appropriate $\tilde{n}$ such that

$$\frac{n+1}{n+\tilde{n}} \leqslant c, \tag{1}$$

$$\frac{n+2}{n+\tilde{n}} > c. \tag{2}$$

Due to Eq. (2), $\tilde{n} \leqslant (n + 2 - cn)/c = \mathrm{O}(n)$. Moreover, the number of commodities $\tilde{m} = \mathrm{O}(\tilde{n}^2) = \mathrm{O}(n^2)$. Now we have constructed a new auction (A', w') whose size is a polynomial of n. Clearly, $S(A', w') = S(A, w) + 2 \leqslant n + 2$.

If $R(A', w') > c$, there must be n satisfied bidders in (A, w) due to Eqs. (1) and (2). Hence the auction admits a Walrasian equilibrium. On the other hand, if $R(A', w') \leqslant c$, we can infer that $S(A, w) < n$, which implies that the Walrasian equilibrium does not exist in the auction (A, w). Now the reduction is complete. □

4. Upper and lower bounds of $R(A, w)$

We have seen from Lemma 2 that $R(A, w)$ can be as small as possible if the number of commodities is much greater than the number of bidders. In the next proposition, we show that even when the number of bidders and the number of commodities are approximately equal, we still cannot promise any satisfactory lower bound for $R(A, w)$.

Proposition 1. *For any prime number p, we can construct a single-minded auction* (A, w) *in which there are* $(p^2 + p + 1)$ *bidders and* $(p^2 + p + 1)$ *commodities such that* $S(A, w) = (p + 1)$.

Proof. The idea of the proof is to construct an auction such that there is at most one winner and p satisfied losers in any feasible allocation-pricing scheme.

In the language of matrix, we are seeking (A, w) with the following properties:

- $\langle a^i, a^j \rangle = 1$ for all $1 \leqslant i, j \leqslant n$, $i \neq j$, i.e., every two bidders share one common item;
- the number of 1's in each row of A is p, i.e., each item is interested by p bidders;
- $w = e$, i.e., every bidder's budget is 1.

The first property ensures that there is at most one winner in any feasible allocation. Assume bidder 1 wins and $a^1 = (1, \ldots, 1, 0, \ldots, 0)^{\mathrm{T}}$ in which the first l entries are 1's. Assume P is the price vector which maximizes the number of satisfied bidders. By the condition of market clearing price, only the first l items of P are nonzero values. For a loser j, if $\langle a^1, a^j \rangle = 0$, it cannot be satisfied because $w_j - \langle P, a^j \rangle = w_j = 1 > 0$. Hence if bidder j is a satisfied loser, we must have $\langle a^1, a^j \rangle = 1$. Assume the first entry of a^j is 1 and 2 to l entries are 0's. By the definition of a satisfied loser, we have $\langle P, a^j \rangle = w_j = 1$ which implies $P = (1, 0, \ldots, 0)^{\mathrm{T}}$. However, by the second property, there are at most p losers who can be satisfied under this price vector because only $p + 1$ bidders are interested in the first item. Therefore, these three properties guarantee that $S(A, w) = p + 1$.

Following the above discussions, we need only construct a 0–1 matrix A of dimension $(p^2+p+1) \times (p^2+p+1)$ which satisfies: (1) the inner product of any two columns is 1; (2) every row contains exactly $(p + 1)$ 1's. However, it is a well-known result in combinatorial designs (e.g. [1, p. 100]). □

In the rest of this section, we will study the lower bound of R by a greedy algorithm. And then we will improve the bound by a more complicated analysis.

The following greedy algorithm gives us the first lower bound estimation of R when the size of each bidder's basic bundle is uniformly bounded by a constant. For the ease of description, we introduce some notations. Let $N^i(A)$ and $N_j(A)$ be the number of 1's in ith column and jth row of a 0–1 matrix A, respectively. Denote by $I_j(A)$ the set of bidders whose basic bundles contain the jth commodity. For a set of bidders E, define *MAX_BIDDER*(E) as the one whose bid is highest.

Data : A single-minded auction (A, w)
Result : An allocation and price (x, p)
Set $k = 0$, $A_k = A$;
while A_k *is not a zero matrix* **do**
  Set $j_k = \text{argmax}_j \left\{ N_j(A_k) \right\}$, i.e., find the commodity which attracts most bidders;
  Set $i_k = \mathit{MAX\_BIDDER}(I_{j_k}(A_k))$;
  Set $x_{i_k} = 1$, i.e., allocate to i_k his basic bundle;
  Set p_{j_k}, the price of commodity j_k, to be w_{i_k}, the budget of bidder i_k;
  Remove all bidders whose basic bundle conflicts bidder i_k's basic bundle, i.e., set those columns which intersects a^{i_k} to be zero. Denote the updated matrix as A_{k+1};
  k=k+1
end

Algorithm 1. A greedy algorithm

Proposition 2. *The algorithm ends with at least n/Δ satisfied bidders where Δ is the bound of the size of bidders' basic bundles.*

Proof. Assume in the kth iteration, item j_k and bidder i_k are selected. Without loss of generality, let $j_k = 1$ and i_k's basic bundle be $\left\{\omega_1, \omega_2, \ldots, \omega_{\delta_k}\right\}$. In this iteration of the algorithm; we remove $I_1(A), I_2(A), \ldots, I_{\delta_k}(A)$ from the auction. Due to the pricing scheme, all the bidders in $I_1(A)$ will be satisfied. Moreover, by the greedy step, we have $|I_1(A)| \geqslant \max\left\{|I_2(A)|, \ldots, |I_{\delta_k}(A)|\right\}$. Hence at least $1/\delta_k$ of them can be satisfied. Then the assumption $\delta_k \leqslant \Delta$ yields the approximation bound. □

Proposition 2 naturally implies $R(A, w) \geqslant 1/m$. The following theorem improves the bound to $2/m$.

Theorem 2. *Let (A, w) be a single-minded auction in which n is the number of bidders and m is the number of commodities, then $R(A, w) \geqslant 2/m$.*

Proof. We divide the bidders into two categories: those who are interested in a single item and those who are interested in multiple items. Define

$$I_{=1} = \left\{ 1 \leqslant i \leqslant n \,\middle|\, \sum_{k=1}^{m} a_k^i = 1 \right\}$$

and

$$I_{\geqslant 2} = \left\{ 1 \leqslant i \leqslant n \,\middle|\, \sum_{k=1}^{m} a_k^i \geqslant 2 \right\},$$

for these two categories, respectively.

Next, we construct m collections of bidders:

$$C_j = \left\{ i \in I_{=1} | a^i_{j+1} = 1 \text{ or } a^i_{j+2} = 1 \right\} \cup \left\{ i \in I_{\geqslant 2} | a^i_j = 1 \right\}, \quad j = 1, 2, \ldots, m.$$

We take module n if the subscript runs out of the range $[1, n]$. Clearly, $\sum_{j=1}^{m} |C_j| \geqslant 2n$ since every bidder is counted at least twice. Then there exists a collection C_j such that $|C_j| \geqslant 2n/m$. Without loss of generality, assume $j = 1$.

To complete the proof of the theorem, we prove that all bidders in C_1 can be satisfied under some feasible allocation-pricing scheme.

Split C_1 into the union of six subsets:

$$\begin{aligned}
I_{\{1\}} &= \left\{ i \in C_1 | a^i_1 = 1, a^i_2 = a^i_3 = 0 \right\}, \\
I_{\{2\}} &= \left\{ i \in C_1 | a^i_1 = a^i_3 = 0, a^i_2 = 1 \right\}, \\
I_{\{3\}} &= \left\{ i \in C_1 | a^i_1 = a^i_2 = 0, a^i_3 = 1 \right\}, \\
I_{\{1,2\}} &= \left\{ i \in C_1 | a^i_1 = a^i_2 = 1, a^i_3 = 0 \right\}, \\
I_{\{1,3\}} &= \left\{ i \in C_1 | a^i_1 = a^i_3 = 1, a^i_2 = 0 \right\}, \\
I_{\{1,2,3\}} &= \left\{ i \in C_1 | a^i_1 = a^i_2 = a^i_3 = 1 \right\}.
\end{aligned}$$

For a subset I_{B} of the above six, denote by i_{B} the bidder with the highest bid in I_{B} and w_{B} for his budget. Define $\bar{w}_{\{1,2\}} = w_{\{1,2\}} - w_{\{2\}}$, $\bar{w}_{\{1,3\}} = w_{\{1,3\}} - w_{\{3\}}$, $\bar{w}_{\{1,2,3\}} = w_{\{1,2,3\}} - w_{\{2\}} - w_{\{3\}}$. Then set the price of ω_1, ω_2 and ω_3 by

$$\begin{aligned}
p_1 &= \max \left\{ w_{\{1\}}, \bar{w}_{\{1,2\}}, \bar{w}_{\{1,3\}}, \bar{w}_{\{1,2,3\}} \right\}, \\
p_2 &= w_{\{2\}}, \\
p_3 &= w_{\{3\}}.
\end{aligned}$$

If $p_1 = w_{\{1\}}$, then sell to $i_{\{1\}}$, $i_{\{2\}}$ and $i_{\{3\}}$ their basic bundles, respectively,
If $p_1 = w_{\{1,2\}}$, then sell to $i_{\{1,2\}}$ and $i_{\{3\}}$ their basic bundles, respectively,
If $p_1 = w_{\{1,3\}}$, then sell to $i_{\{1,3\}}$ and $i_{\{2\}}$ their basic bundles, respectively,
If $p_1 = w_{\{1,2,3\}}$, then sell to $i_{\{1,2,3\}}$ his basic bundle.

By the construction of the price assignment, we can ensure that every bidder in C_1 is satisfied. □

Remark 1. We can construct an auction (A, w) with three commodities such that $R(A, w) = 2/3$ (e.g. Example 1). Therefore, the bound in Theorem 2 is tight when $m = 3$.

5. Conclusion and discussion

We define a ratio R in single-minded auctions to provide an alternative for the scenario where Walrasian equilibrium does not exist. We give a general lower bound for R by a constructive proof which can yield at least $2n/m$ satisfied bidders. We also give several upper bounds by illustrating some interesting instances (Lemma 2 and Proposition 1). Hinted by these two instances, we conjecture that the general lower bound can be raised from $2/m$ to $\Omega(1/\sqrt{m})$.

In reality, we may need to consider weighted satisfaction problem to emphasize the different importance of bidders. Furthermore, we may also need to extend the problem to general combinatorial auctions rather than single-minded auctions.

References

[1] I. Anderson, Combinatorial Designs and Tournaments, Oxford University Press, 1997.
[2] S. Bikhchandani, J.W. Mamer, Competitive equilibrium in an exchange economy with indivisibilities, J. Econom. Theory 74 (1997) 385–413.
[3] L. Chen, X. Deng, Q. Fang, F. Tian, Majority equilibrium for public facility allocation, Lecture Notes in Comput. Sci. 2697 (2003) 435–444 preliminary version.
[4] N. Chen, X. Deng, X. Sun, On complexity of single-minded auction, http://www.cs.cityu.edu.hk/deng/papers/single.pdf
[5] G. Demange, Spatial models of collective choice, in: J.F. Thisse, H.-G. Holler (Eds.), Locational Analysis of Public Facilities, North-Holland, Amsterdam, 1983, pp. 153–182.
[6] X. Deng, C. Papadimitriou, S. Safra, On the complexity of price equilibria, J. Comput. Syst. Sci. 67 (2) (2003) 311–324.
[7] F. Gul, E. Stacchetti, Walrasian equilibrium with gross substitutes, J. Econom. Theory 87 (1999) 95–124.
[8] P. Hansen, J.F. Thisse, Outcomes of voting and planning: condorcet, weber and rawls locations, J. Publ. Econom. 16 (1981) 1–15.
[9] A.S. Kelso, V.P. Crawford, Job matching, coalition formation, and gross substitutes, Econometrica 50 (1982) 1483–1504.
[10] N. Nisan, Bidding and allocation in combinatorial auctions, in: Proc. ACM Conf. on Electronic Commerce, 1999, pp. 1–12.
[11] A. Postlewaite, D. Schmeidler, Approximate Walrasian equilibria and nearby economics, Internat. Econom. Rev. 22 (1) (1981) 105–111.
[12] S. De Vries, R. Vohra, Combinatorial auctions: a survey, INFORMS J. Comput. 15 (3) (2003) 284–309.