

Removable edges in a cycle of a 4-connected graph<sup>☆</sup>Jichang Wu<sup>a</sup>, Xueliang Li<sup>b</sup>, Lusheng Wang<sup>c</sup><sup>a</sup>*School of Mathematics and System Science, Shandong University, 27, Shanda South Road, Jinan, Shandong 250100, PR China*<sup>b</sup>*Center for Combinatorics, Nankai University, Tianjin 300071, PR China*<sup>c</sup>*Department of Computer Science, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, PR China*

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**Abstract**

Let  $G$  be a 4-connected graph. For an edge  $e$  of  $G$ , we do the following operations on  $G$ : first, delete the edge  $e$  from  $G$ , resulting in the graph  $G - e$ ; second, for all the vertices  $x$  of degree 3 in  $G - e$ , delete  $x$  from  $G - e$  and then completely connect the 3 neighbors of  $x$  by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by  $G \ominus e$ . If  $G \ominus e$  is still 4-connected, then  $e$  is called a *removable edge* of  $G$ . In this paper, we investigate the problem on how many removable edges there are in a cycle of a 4-connected graph, and give examples to show that our results are in some sense the best possible.

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**Keywords:** 4-Connected graph; Removable edge; Edge-vertex-cut fragment; Edge-vertex-cut atom**1. Introduction**

All graphs considered here are simple and finite. For notations and terminology not given here, we refer the reader(s) to [2]. In this paper we shall study the removable edges in a cycle of a 4-connected graph. First of all, we give the definition of a removable edge for a 4-connected graph. Let  $G$  be a 4-connected graph and  $e$  an edge of  $G$ . Consider the graph  $G - e$  obtained by deleting the edge  $e$  from  $G$ . If  $G - e$  has vertices of degree 3, we do the following operations on  $G - e$ . For all vertices  $x$  of degree 3 in  $G - e$ , delete  $x$  from  $G - e$  and then completely connect the three neighbors of  $x$  by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by  $G \ominus e$ . Note that if there is no vertex of degree 3 in  $G - e$ , then  $G \ominus e$  is simply the graph  $G - e$ .

**Definition 1.1.** For a 4-connected graph  $G$  and an edge  $e$  of  $G$ , if  $G \ominus e$  is still 4-connected, then the edge  $e$  is called *removable*; otherwise, it is called *unremovable*. The set of all removable edges of  $G$  is denoted by  $E_R(G)$ ; whereas the set of unremovable edges of  $G$  is denoted by  $E_N(G)$ .

**Definition 1.2.** A 2-cyclic graph  $G$  of order  $n$  is defined to be the square of the cycle  $C_n$ , namely,  $G$  can be obtained from  $C_n$  by adding edges between all pairs of vertices of  $C_n$  which are at distance 2 in  $C_n$ .

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The aim to introduce the concept of removable edges in 4-connected graphs is to find new methods to construct 4-connected graphs and to prove some properties of 4-connected graphs inductively. Yin [7], proved that there always exist removable edges in 4-connected graphs  $G$  unless  $G$  is a 2-cyclic graph of order 5 or 6. He showed that a 4-connected graph can be obtained from a 2-cyclic graph by the following four operations: (i) adding edges, (ii) splitting vertices, (iii) adding vertices and removing edges, and (iv) extending vertices. He also obtained a lower bound for the number of removable edges and contractible edges in a 4-connected graph  $G$ . In this paper, we shall investigate how many removable edges there are in a cycle of a 4-connected graph  $G$ , and give examples to show that our results are the best possible in some sense.

For convenience we introduce the following notations. Without a specific statement, in the sequel  $G$  always denotes a 4-connected graph. The vertex set and edge set of  $G$  is denoted, respectively, by  $V(G)$  and  $E(G)$ . The order and size of  $G$  is denoted, respectively, by  $|G|$  and  $|E(G)|$ . For  $x \in V(G)$ , we simply write  $x \in G$ . The neighborhood of  $x \in G$  is denoted by  $\Gamma_G(x)$  and the degree of  $x$  is denoted by  $d(x)$ . If  $x$  and  $y$  are the two end-vertices of an edge  $e$ , we write  $e = xy$ . For a nonempty subset  $F$  of  $E(G)$ , or  $N$  of  $V(G)$ , the induced subgraph by  $F$  or  $N$  in  $G$  is denoted by  $[F]$  or  $[N]$ . Let  $A, B \subset V(G)$  such that  $A \neq \emptyset \neq B$  and  $A \cap B = \emptyset$ , define  $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$ . If  $H$  is a subgraph of  $G$ , we say that  $G$  contains  $H$ . For a subset  $S$  of  $V(G)$ ,  $G - S$  denotes the graph obtained by deleting all the vertices in  $S$  from  $G$  together with all the incident edges. If  $G - S$  is disconnected, we say that  $S$  is a vertex-cut of  $G$ . If  $|S| = s$  for such an  $S$ , we say that  $S$  is an  $s$ -vertex-cut. For  $e = xy \in E(G)$  and  $S \subset V(G)$  such that  $|S| = 3$ , if  $G - e - S$  has exactly two (connected) components, say  $A$  and  $B$ , such that  $|A| \geq 2$  and  $|B| \geq 2$ , then we say that  $(e, S)$  is a *separating pair* and  $(e, S; A, B)$  is a *separating group*, in which  $A$  and  $B$  are called the *edge-vertex-cut fragments*. If, moreover,  $|A| = 2$ , then  $A$  is called an *edge-vertex-cut atom*. For an edge-vertex-cut atom  $A$ , let  $A = \{x, z\}$  and  $S = \{a, b, c\}$ , if  $ax, bx \in E(G)$ ,  $cx \notin E(G)$ , then  $A$  is called a *1-edge-vertex-cut atom*, whereas if  $ax, bx, cx \in E(G)$ , then  $A$  is called a *2-edge-vertex-cut atom*. It is easy to see that if  $A$  is an edge-vertex-cut atom, then  $A$  is either a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom. Let  $E_0 \subset E_N(G)$  such that  $E_0 \neq \emptyset$  and let  $(xy, S; A, B)$  be a separating group of  $G$  such that  $x \in A$  and  $y \in B$ . If  $xy \in E_0$ , then  $A$  and  $B$  are called  *$E_0$ -edge-vertex-cut fragments*. An  $E_0$ -edge-vertex-cut fragment is called an  *$E_0$ -edge-vertex-cut end-fragment* of  $G$  if it does not contain any other  $E_0$ -edge-vertex-cut fragment of  $G$  as a proper subset. It is easy to see that any  $E_0$ -edge-vertex-cut fragment of  $G$  contains such an end-fragment. Similarly, if  $|A| = 2$ , then  $A$  is called an  *$E_0$ -edge-vertex-cut atom*.

## 2. Some known results

In the sequel, we shall use the following results on the existence of removable edges in 4-connected graphs, which were obtained by Yin [7].

**Theorem 2.1.** *Let  $G$  be a 4-connected graph with  $|G| \geq 7$ . An edge  $e$  of  $G$  is unremovable if and only if there is a separating pair  $(e, S)$ , or a separating group  $(e, S; A, B)$  in  $G$ .*

**Theorem 2.2.** *Let  $G$  be a 4-connected graph with  $|G| \geq 8$  and let  $(xy, S; A, B)$  be a separating group of  $G$  such that  $x \in A$ ,  $y \in B$  and  $|A| \geq 3$ . Then, every edge in  $\{[x], S\}$  is removable.*

**Corollary 2.3.** *Let  $G$  be a 4-connected graph with  $|G| \geq 8$ . Then, every 3-cycle of  $G$  contains at least one removable edge.*

**Theorem 2.4.** *Let  $G$  be a 4-connected graph with  $|G| \geq 7$ . If for an unremovable edge  $xy$ , i.e.,  $xy \in E_N(G)$ , there is a separating group  $(xy, S; A, B)$ , then all the edge in  $E([S])$  are removable, i.e.,  $E([S]) \subset E_R(G)$ .*

## 3. Notations and terminology for subgraphs with special structures

For convenience we introduce the following definitions for subgraphs of  $G$  with special structures.

**Definition 3.1.** Let  $G$  be a 4-connected graph and  $H$  a subgraph of  $G$  such that  $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$  and  $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$ . If  $H$  satisfies the following conditions

- (i)  $d(a) = d(x_i) = 4$  for  $i = 1, 2, 3, 4$ ,
  - (ii)  $ax_1, ax_2, ax_3, ax_4 \in E_N(G)$  and  $x_1x_2, x_2x_3, x_3x_4, x_4x_1 \in E_R(G)$ ,
- then  $H$  is called a *helm*. The vertices  $a, x_i$  for  $i = 1, 2, 3, 4$  of a helm  $H$  are called *inner vertices* of  $H$ .

**Definition 3.2.** Let  $G$  be a 4-connected graph and  $H$  a subgraph of  $G$  such that  $V(H) = \{a, b, x_1, x_2, \dots, x_{l+3}\}$  and  $E(H) = \{x_1x_2, x_2x_3, \dots, x_{l+2}x_{l+3}, ax_2, ax_3, \dots, ax_{l+2}, bx_2, bx_3, \dots, bx_{l+2}\}$ , where  $l \geq 1$ . If  $H$  satisfies the following conditions

- (i)  $x_i x_{i+1} \in E_N(G)$  for  $i = 1, 2, \dots, l+2$ ,
- (ii)  $ax_j, bx_j \in E_R(G)$  for  $j = 2, 3, \dots, l+2$ ,
- (iii)  $d(x_j) = 4$  for  $j = 2, 3, \dots, l+2$ ,

then  $H$  is called an *l-bi-fan*.

An *l-bi-fan*  $H$  is said to be *maximal* if  $\Gamma_G(x_1) \neq \{a, b, x_2, u\}$  and  $\Gamma_G(x_{l+3}) \neq \{a, b, x_{l+2}, v\}$  for any  $u, v \in G$ . The vertices of an *l-bi-fan* or a maximal *l-bi-fan*  $H$  satisfying the condition (iii) are called *inner vertices* of  $H$ .

**Definition 3.3.** Let  $G$  be a 4-connected graph and  $H$  a subgraph of  $G$  such that  $V(H) = \{x_1, x_2, \dots, x_{l+2}, y_1, y_2, \dots, y_{l+2}\}$  and  $E(H) = E_1(H) \cup E_2(H)$  where  $E_1(H) = \{x_1x_2, x_2x_3, \dots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \dots, y_{l+1}y_{l+2}\}$  and  $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \dots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$ . Then,  $H$  is called an *l-belt* if the following conditions are satisfied

- (i)  $E_1(H) \subset E_N(H)$  and  $E_2(H) \subset E_R(H)$ ,
- (ii)  $d(x_i) = d(y_j) = 4$  for  $i = 2, 3, \dots, l+1$ ;  $j = 2, 3, \dots, l+1$ .

An *l-belt*  $H$  is said to be *maximal* if  $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$  and  $\Gamma_G(x_{l+2}) \neq \{x_{l+1}, y_{l+1}, y_{l+2}, v\}$  for any  $u, v \in G$ . The vertices of an *l-belt* or a maximal *l-belt*  $H$  satisfying condition (ii) are called *inner vertices* of  $H$ .

**Definition 3.4.** Let  $G$  be a 4-connected graph and  $H$  a subgraph of  $G$  such that  $V(H) = \{x_1, x_2, \dots, x_{l+2}, x_{l+3}, y_1, y_2, \dots, y_{l+2}\}$  and  $E(H) = E_1(H) \cup E_2(H)$ , where  $E_1(H) = \{x_1x_2, x_2x_3, \dots, x_{l+1}x_{l+2}, x_{l+2}x_{l+3}, y_1y_2, y_2y_3, \dots, y_{l+1}y_{l+2}\}$  and  $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \dots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}, x_{l+2}y_{l+2}\}$ . Then,  $H$  is called an *l-co-belt* if the following conditions are satisfied

- (i)  $E_1(H) \subset E_N(H)$  and  $E_2(H) \subset E_R(H)$ ,
- (ii)  $d(x_i) = d(y_j) = 4$  for  $i = 2, 3, \dots, l+1, l+2$ ;  $j = 2, 3, \dots, l+1$ .

An *l-co-belt*  $H$  is said to be *maximal* if  $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$  and  $\Gamma_G(y_{l+2}) \neq \{x_{l+2}, y_{l+1}, x_{l+3}, v\}$  for any  $u, v \in G$ . The vertices of an *l-co-belt* or a maximal *l-co-belt*  $H$  satisfying condition (ii) are called *inner vertices* of  $H$ .

**Definition 3.5.** Let  $G$  be a 4-connected graph and  $H$  a subgraph of  $G$  such that  $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$  and  $E(H) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$ . Then,  $H$  is called a *W-framework* if the following conditions are satisfied

- (i)  $x_i x_{i+1} \in E_N(G)$  for  $i = 1, 2$ ,
- (ii)  $d(x_2) = d(y_2) = d(y_3) = 4$ ,
- (iii)  $y_2y_3, x_1y_2, x_2y_2, x_2y_3, x_3y_3 \in E_R(G)$ .

The vertex  $x_2$  of a *W-framework*  $H$  is called the *inner vertex* of  $H$ .

**Definition 3.6.** Let  $G$  be a 4-connected graph and  $H$  a subgraph of  $G$  such that  $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$  and  $E(H) = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$ . Then,  $H$  is called a *W'-framework* if the following conditions are satisfied

- (i)  $x_i x_{i+1} \in E_N(G)$  for  $i = 1, 2$ ,
- (ii)  $d(x_2) = d(x_3) = d(y_2) = d(y_3) = 4$  and  $d(x_1) \geq 5$ ,
- (iii)  $y_2y_3, x_1y_2, x_2y_3, x_3y_3, x_1x_3 \in E_R(G)$ ,  $x_2y_2 \in E_N(G)$ .

The vertices  $x_2, x_3$  of a *W'-framework*  $H$  are called *inner vertices* of  $H$ .

After we have done the above preparations, we can state and prove our main results in the next section.

#### 4. The main results

In this section we shall consider the problem on how many removable edges there are in a cycle of a 4-connected graph  $G$ . Before we give our main results, we need to show some lemmas.

**Lemma 4.1.** Let  $G$  be a 4-connected graph,  $(xy, S; A, B)$  be a separating group of  $G$  such that  $x \in A, y \in B, S = \{a, b, c\}$  and  $A$  be a 1-edge-vertex atom, say,  $A = \{x, z\}$ . Then, one of the following conclusions holds:

- (i)  $ax, bx, zx \in E_R(G)$ .
- (ii)  $ax \in E_N(G), d(x) = d(z) = 4, bx, zx, az \in E_R(G), zc \in E_N(G)$ .
- (iii)  $ax \in E_N(G), ay \in E_R(G)$ . And, if  $d(a)=4, d(y) \geq 5$ , then  $az, zb, zx, by \in E_R(G), bx \in E_N(G)$ . If  $d(a) \geq 5, d(y)=4$ , then  $by, bx, bz, az \in E_R(G), zx \in E_N(G)$ . If  $d(a)=d(y)=4$ , then  $az, bz, by \in E_R(G), bx, zx \in E_N(G)$ . If  $d(a) \geq 5, d(y) \geq 5$ , then  $az, zx, bx, by \in E_R(G)$ .
- (iv)  $ax, bx, ac, bc \in E_R(G), zx, zc \in E_N(G), \{za, zb\} \cap E_N(G) \neq \emptyset, d(x) = d(c) = d(z) = 4$ . If  $za \in E_N(G)$ , then the following conclusion holds:  $d(b)=4$ , and if  $d(a)=4$ , then  $bz \in E_N(G)$ ; if  $d(a) \geq 5$ , then  $bz \in E_R(G)$  holds. If  $bz \in E_N(G)$ , then the following conclusion holds:  $d(a) = 4$ , and if  $d(b) = 4$ , then  $az \in E_N(G)$ ; if  $d(b) \geq 5$ , then  $az \in E_R(G)$ .
- (v)  $ax, bx, az, bz \in E_R(G), xz \in E_N(G), d(x) = d(z) = 4$ .
- (vi)  $bx \in E_N(G), by \in E_R(G)$ . And, if  $d(b)=4, d(y) \geq 5$ , then  $bz, za, zx, ay \in E_R(G), ax \in E_N(G)$ . If  $d(b) \geq 5, d(y)=4$ , then  $ay, ax, az, bz \in E_R(G), zx \in E_N(G)$ . If  $d(b)=d(y)=4$ , then  $bz, az, ay \in E_R(G), ax, zx \in E_N(G)$ . If  $d(b) \geq 5, d(y) \geq 5$ , then  $bz, zx, ax, ay \in E_R(G)$ .
- (vii)  $bx \in E_N(G), d(x) = d(z) = 4, ax, zx, bz \in E_R(G), zc \in E_N(G)$ .

**Proof.** If  $ax, bx, zx \in E_R(G)$ , then the conclusion (i) holds. So, we may assume that  $\{ax, bx, zx\} \cap E_N(G) \neq \emptyset$ . Next, we will distinguish the following cases to proceed the proof.

Case 1:  $ax \in E_N(G)$ .

Then, we take the corresponding separating group  $(ax, T; C, D)$  such that  $x \in C, a \in D$ , and so,  $x \in A \cap C, y \in B \cap (C \cup T)$ . Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T),$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (S \cap D),$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T),$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S).$$

Subcase 1.1:  $y \in B \cap C$ .

Since  $|A| = 2$  and  $A$  is a connected subgraph of  $G$ , we have that  $A \cap D = \emptyset$ . First, we claim that  $A \cap T \neq \emptyset$ . Otherwise,  $A \cap T = \emptyset$ , and so  $|A \cap C| = 2$ . Since  $a \in S \cap D$ , we have that  $|X_1| \leq 2$ . Then,  $X_1 \cup \{x\}$  is a vertex-cut of  $G$  with cardinality less than 4, a contradiction. Hence,  $A \cap T = \{z\}$ . Second, we claim that  $S \cap T = \emptyset$ . Otherwise,  $S \cap T \neq \emptyset$ , and a contradiction will be deduced as follows: if  $B \cap T = \emptyset$ , since  $B$  is a connected subgraph of  $G$ , then we have that  $B \cap D = \emptyset$ . Then,  $B = B \cap C$ , and so  $|S \cap T| = 2$ . Noticing that  $a \in S \cap D$  and  $|S| = 3$ , we have that  $S \cap C = \emptyset$ . From  $|B| \geq 2$  we know that  $|B \cap C| \geq 2$ . Then, it is easy to see that  $\{y\} \cup (S \cap T)$  is a vertex-cut of  $G$  with cardinality less than 4, a contradiction. So,  $B \cap T \neq \emptyset$ , and so  $|S \cap T| = 1$ . Noticing that  $|T| = 3$ , we have that  $|B \cap T| = 1$ . Since  $X_4$  is a vertex-cut of  $G - xy$ , we have that  $|X_4| \geq 3$ , and so,  $|S \cap C| \geq 1$ . Since  $S \cap D \neq \emptyset$ , by noticing that  $|S| = 3$ , we have that  $|S \cap D| = 1$ , i.e.,  $S \cap D = \{a\}$ . Note that  $|X_3| = 3$ . Since  $G$  is 4-connected, we have that  $B \cap D = \emptyset$ . Hence,  $D = \{a\}$ , which contradicts to that  $|D| \geq 2$ . Therefore,  $S \cap T = \emptyset$ . Note that  $|B \cap T| = 2$ . If  $|S \cap D| = 1$ , by a similar argument we can get that  $D = \{a\}$ , a contradiction. So,  $|S \cap D| \geq 2$ . Since  $|X_4| \geq 3$ , we have that  $|S \cap C| \geq 1$ . Therefore,  $|S \cap C| = 1$  and  $|S \cap D| = 2$ . Since  $bx \in E(G)$ , obviously we have  $b \in X_1$ , and so  $S \cap C = \{b\}$ . Then,  $S \cap D = \{a, c\}$ ,  $\Gamma_G(x) = \{a, b, y, z\}$ ,  $\Gamma_G(z) = \{x, a, b, c\}$ . We claim that  $xz \in E_R(G)$ . Otherwise,  $xz \in E_N(G)$ , and we take the corresponding separating group  $(xz, S'; A', B')$  such that  $x \in A', z \in B'$ . Since  $xzax$  is a 3-cycle of  $G$ , we have that  $a \in S'$  and  $ax \in E_N(G)$ . From Theorem 2.2 we know that  $|A'| = 2$ , say  $A' = \{x, v_1\}$ . Then, we have that  $axv_1a$  is a 3-cycle of  $G$  and  $v_1 \neq z$ , which is impossible to hold in  $G$ , and so,  $xz \in E_R(G)$ . We claim that  $az \in E_R(G)$ . Otherwise,  $az \in E_N(G)$ , and we take the corresponding separating group  $(az, S'; A', B')$  such that  $a \in A', z \in B'$ . Obviously,  $x \in S'$ . Since  $ax \in E_N(G)$ , from Theorem 2.2 we have that  $|A'| = 2$ , say  $A' = \{a, v_1\}$ . Then,  $axv_1a$  is a 3-cycle of  $G$  and  $v_1 \neq z$ , which is impossible to hold in  $G$ , and so,  $az \in E_R(G)$ . Let  $S' = \{x\} \cup (B \cap T)$ ,  $A' = C \cap (B \cup S)$ ,  $B' = G - bz - S' - A'$ , then  $(bz, S'; A', B')$  is a separating group of  $G$ , and so  $bz \in E_N(G)$ . We claim that  $bx \in E_R(G)$ . Otherwise,  $bx \in E_N(G)$ , and we take the corresponding separating group  $(bx, S'; A', B')$  such that  $b \in A', x \in B'$ . Since  $bxzb$  is a 3-cycle of  $G$ , we have that  $z \in S'$ . Since  $bz \in E_N(G)$ , we have that  $|A'| = 2$ , say  $A' = \{b, v_1\}$ . Then,  $bv_1zb$  is a 3-cycle of  $G$ , and  $v_1 \neq x$ , which is impossible to hold in  $G$ , and hence  $bx \in E_R(G)$ . Let  $S_1 = \{a, b, y\}$ , then  $(zc, S_1)$  is a separating pair of  $G$ , and so,  $zc \in E_N(G)$ . Obviously,  $d(x) = d(z) = 4$ . Hence, conclusion (ii) holds.

Subcase 1.2:  $y \in B \cap T$ .

Since  $xy \in E_N(G)$ , from Theorem 2.2 we have that  $|C| = 2$ . If  $|A \cap C| = 2$ , then we have that  $A = A \cap C = C$ . Since  $B \cap T \neq \emptyset \neq S \cap D$ , we have that  $|S \cap T| \leq 2$ . It is easy to see that  $\{x\} \cup X_1$  is a vertex-cut of  $G$  with cardinality less than

4, a contradiction. So,  $A \cap C = \{x\}$ . Since  $A$  and  $C$  are connected subgraphs of  $G$ , we have that  $|S \cap C| = |A \cap T| = 1$  and  $B \cap C = \emptyset = A \cap D$ . We claim that  $S \cap T = \emptyset$ . Otherwise,  $|S \cap T| = 1$ , and so  $|B \cap T| = 1$ . Note that  $|X_3| = 3$ . Since  $G$  is 4-connected, we have that  $B \cap D = \emptyset$ , and so  $B = B \cap T = \{y\}$ , which contradicts to that  $|B| \geq 2$ . Therefore,  $S \cap T = \emptyset$ , and so  $|B \cap T| = |S \cap D| = 2$ . From  $\Gamma_G(x) = \{z, b, a, y\}$  we know that  $S \cap C = \{b\}$ , and so  $S \cap D = \{a, c\}$ ,  $A \cap T = \{z\}$ . Let  $B \cap T = \{u, y\}$ . Next we will discuss the following subcases.

*Subsubcase 1.2.1:* If  $ay \notin E(G)$ , we claim that  $xz \in E_R(G)$ . Otherwise,  $xz \in E_N(G)$ , and we take the corresponding separating group  $(xz, S'; A', B')$  such that  $z \in A'$ ,  $x \in B'$ . Since  $axza$  is a 3-cycle of  $G$ , we have that  $a \in S'$ . Since  $ax \in E_N(G)$ , from Theorem 2.2 we have that  $|B'| = 2$ , say  $B' = \{x, v_1\}$ . Then,  $axv_1a$  is a 3-cycle of  $G$ . However,  $ay \notin E(G)$  and  $v_1 \neq z$ , which is impossible to hold in  $G$ . Hence,  $xz \in E_R(G)$ . By symmetry, we can show that  $bx \in E_R(G)$ . We claim that  $az \in E_R(G)$ . Otherwise,  $az \in E_N(G)$ , and we take the corresponding separating group  $(az, S'; A', B')$  such that  $a \in A'$ ,  $z \in B'$ . Since  $azxa$  is a 3-cycle of  $G$ , we have that  $x \in S'$ . Since  $ax \in E_N(G)$ , we have that  $|A'| = 2$ , say  $A' = \{a, v_1\}$ . Then,  $axv_1a$  is a 3-cycle of  $G$ , an analogous argument can lead to a contradiction. So,  $az \in E_R(G)$ . By symmetry, we have that  $by \in E_R(G)$ . Let  $S' = \{a, b, y\}$ . Obviously,  $(zc, S')$  is a separating pair of  $G$ , and so  $zc \in E_N(G)$ . Hence, the conclusion (ii) holds.

*Subsubcase 1.2.2:* If  $ay \in E(G)$ , then from Corollary 2.3 we know that  $ay \in E_R(G)$ . Then, we consider the following cases.

- (1) If  $d(a) \geq 5$  and  $d(y) \geq 5$ , we claim that  $xz \in E_R(G)$ . Otherwise,  $xz \in E_N(G)$ , and we take the corresponding separating group  $(xz, S'; A', B')$  such that  $x \in A'$ ,  $z \in B'$ . Since  $axza$  is a 3-cycle of  $G$ , we have that  $a \in S'$ . Since  $ax \in E_N(G)$ , from Theorem 2.2 we know that  $|A'| = 2$ , say  $A' = \{x, v_1\}$ . Then,  $axv_1a$  is a 3-cycle of  $G$ . Noticing that  $d(v_1) = 4$  and  $d(y) \geq 5$ , we have that  $v_1 \neq y$ , which is impossible to hold in  $G$ . Hence,  $xz \in E_R(G)$ . By symmetry, we can show that  $bx \in E_R(G)$ . We claim that  $az \in E_R(G)$ . Otherwise,  $az \in E_N(G)$ , and we take the corresponding separating group  $(az, S'; A', B')$ . Obviously,  $x \in S'$ , and an analogous argument can lead to a contradiction. So,  $az \in E_R(G)$ . By symmetry, we have that  $by \in E_R(G)$ . Hence, the conclusion (iii) holds.
- (2) If  $d(a) = 4$  and  $d(y) \geq 5$ , we let  $\Gamma_G(a) = \{x, y, z, v\}$ . Let  $A' = \{a, x\}$ ,  $S' = \{v, z, y\}$ ,  $B' = G - bx - S' - A'$ , then  $(bx, S'; A', B')$  is a separating group of  $G$ , and so  $bx \in E_N(G)$ . We claim that  $bz \in E_R(G)$ . Otherwise,  $bz \in E_N(G)$ , and we take the corresponding separating group  $(bz, S'; A', B')$  such that  $b \in A'$ ,  $z \in B'$ . Noticing that  $bzxb$  is a 3-cycle of  $G$ , we have  $x \in S'$ . Since  $bx \in E_N(G)$ , from Theorem 2.2 we have that  $|A'| = 2$ , say,  $A' = \{b, v_1\}$ . Then,  $bxv_1b$  is a 3-cycle of  $G$ . Noticing that  $d(y) \geq 5$  and  $d(v_1) = 4$ , we have that  $v_1 \neq y$ , which is impossible to hold in  $G$ . Therefore,  $bz \in E_R(G)$ . We claim that  $az \in E_R(G)$ . Otherwise,  $az \in E_N(G)$ , and we take the separating group  $(az, S'; A', B')$  such that  $a \in A'$ ,  $z \in B'$ . Obviously,  $x \in S'$ . Since  $ax \in E_N(G)$ , from Theorem 2.2 we have that  $|A'| = 2$ , say  $A' = \{a, v_1\}$ . Then,  $axv_1a$  is a 3-cycle of  $G$  and  $v_1 \neq z$ . Note that  $d(v_1) = 4$ ,  $d(y) \geq 5$ , and so,  $v_1 \neq y$ , which is impossible to hold in  $G$ . So,  $az \in E_R(G)$ . By an analogous argument we can show that  $zx \in E_R(G)$ . We claim that  $by \in E_R(G)$ . Otherwise,  $by \in E_N(G)$ , and we take the separating group  $(by, S'; A', B')$  such that  $b \in A'$ ,  $y \in B'$ . Obviously,  $x \in S'$ . Since  $xy \in E_N(G)$ , from Theorem 2.2 we have that  $|B'| = 2$ , say  $B' = \{y, v_1\}$ . Then,  $xyv_1x$  is a 3-cycle of  $G$ . It is easy to see that this is true only if  $v_1 = a$ . From  $\Gamma_G(a) = \{x, y, z, v\}$  we know that  $S' = \{x, z, v\}$ . Since  $d(y) \geq 5$ , we have  $yz \in E(G)$ , which is impossible to hold in  $G$ . So,  $by \in E_R(G)$ . Hence, the conclusion (iii) holds.
- (3) If  $d(a) \geq 5$  and  $d(y) = 4$ . By an analogous argument used in (2) we can show that conclusion (iii) holds.
- (4) If  $d(a) = d(y) = 4$ , we let  $\Gamma_G(a) = \{x, y, z, v\}$ ,  $A_1 = \{a, x\}$ ,  $S_1 = \{z, y, v\}$ ,  $B_1 = G - bx - S_1 - A_1$ . Then,  $(bx, S_1; A_1, B_1)$  is a separating group of  $G$ , and so  $bx \in E_N(G)$ . By symmetry, we have that  $ax, xy, zx \in E_N(G)$ . From Corollary 2.3 we have that  $az, by, bz \in E_R(G)$ . Hence, conclusion (iii) holds.

If  $bx \in E_N(G)$ , we may employ a similar argument to show that conclusion (vi) or (vii) hold. So, next we may assume that  $ax, bx \in E_R(G)$ .

*Case 2:*  $xz \in E_N(G)$ .

We take the corresponding separating group  $(xz, T; C, D)$  such that  $x \in C$ ,  $z \in D$ . Then, we have that  $x \in A \cap C$ ,  $z \in A \cap D$ . Since  $xzax, xzbx$  are two 3-cycles of  $G$ , we have that  $a, b \in S \cap T$ . Since  $A \cap D = \{z\}$  and  $D$  is a connected subgraph of  $G$  as well as  $|D| \geq 2$ , we can get that  $S \cap D \neq \emptyset$ . Since  $S = \{a, b, c\}$ , we have that  $S \cap D = \{c\}$ . Obviously,  $|B \cap T| = 1$ .

*Subcase 2.1:* If  $az \in E_N(G)$ , from Theorem 2.2 we have that  $|D| = 2$ , and so  $D = \{z, c\}$ . It is easy to see that  $ac, bc \in E(G)$ . From Theorem 2.4 we have that  $ac, bc \in E_R(G)$ . Obviously,  $d(x) = d(c) = d(z) = 4$  and  $\Gamma_G(x) = \{z, b, a, y\}$ . Let  $A_1 = \{x, z\}$ ,  $S_1 = \{y, a, b\}$ ,  $B_1 = G - zc - S_1 - A_1$ , then  $(zc, S_1; A_1, B_1)$  is a separating group of  $G$ , and so  $zc \in E_N(G)$ . We take the separating group  $(az, S'; A', B')$  such that  $a \in A'$ ,  $z \in B'$ . Obviously,  $x \in S'$ . Since  $xz \in E_N(G)$ , we have that  $|B'| = 2$ , say  $B' = \{z, v_1\}$ . Then,  $xzv_1x$  is a 3-cycle of  $G$ , which is true only if  $v_1 = b$ , and so  $d(b) = 4$ . Here, if  $d(a) = 4$ , let  $\Gamma_G(a) = \{x, z, c, v\}$ ,  $A_1 = \{a, z\}$ ,  $S_1 = \{c, x, v\}$  and  $B_1 = G - bz - S_1 - A_1$ . Then  $(bz, S_1; A_1, B_1)$  is a separating group of  $G$ , and so  $bz \in E_N(G)$ . If  $d(a) \geq 5$ , we claim that  $bz \in E_R(G)$ . Otherwise,  $bz \in E_N(G)$ , then we take the corresponding separating group  $(bz, S_1; A_1, B_1)$  such that  $b \in A_1$ ,  $z \in B_1$ . Obviously,  $x \in S_1$ . Since  $xz \in E_N(G)$ , from Theorem 2.2 we have  $|B_1| = 2$ , say  $B_1 = \{z, v_1\}$ . Then  $xv_1zx$  is a 3-cycle of  $G$ . Note that  $d(a) \geq 5$ ,  $d(v_1) = 4$ , and so  $v_1 \neq a$ . Which is impossible to hold in  $G$ . So,  $bz \in E_R(G)$ . Hence, the conclusion (iv) holds.

*Subcase 2.2:* If  $bz \in E_N(G)$ , we may employ a similar argument used in subcase 2.1 to show that conclusion (iv) holds.



Therefore, we may assume that  $az, bz \in E_R(G)$ . Obviously,  $d(x) = d(z) = 4$ , and so conclusion (v) holds. The proof is now complete.  $\square$

From the Lemma 4.1 and its proof, we may get the following corollary.

**Corollary 4.2.** *Let  $G$  be a 4-connected graph and  $(xy, S; A, B)$  be a separating group of  $G$  such that  $x \in A, y \in B, S = \{a, b, c\}$ . Let  $A$  be a 1-edge-vertex-cut atom, say  $A = \{x, z\}$ . If  $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$ , then we have that  $x$  is an inner vertex of one of the following subgraphs in  $G$ : helm,  $l$ -co-belt,  $l$ -belt,  $W'$ -framework,  $W$ -framework or  $l$ -bi-fan.*

**Lemma 4.3.** *Let  $G$  be a 4-connected graph,  $(xy, S; A, B)$  be a separating group of  $G$ , and  $A$  be a 2-edge-vertex-cut atom, say  $A = \{x, z\}$  and  $S = \{a, b, c\}$ . Then,  $ax, bx, cx, xz \in E_R(G)$ .*

**Proof.** By contradiction. We consider the following cases.

- (1) If  $ax \in E_N(G)$ , we take the corresponding separating group  $(ax, T; C, D)$  such that  $x \in C, a \in D$ . Then,  $x \in A \cap C, a \in S \cap D$ . Let  $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$ . Since  $bx, cx \in E(G)$ , we can get that  $b, c \in S \cap (C \cup T)$ , and so  $|S \cap D| = 1$ . We claim that  $A \cap T \neq \emptyset$ . Otherwise,  $A \cap T = \emptyset$ . Since  $|A| = 2$  and  $A$  is a connected subgraph of  $G$ , we have that  $A \cap C = \{x, z\}$ . It is easy to see that  $\{b, c, x\}$  would be a 3-vertex-cut of  $G$ , a contradiction. Therefore,  $A \cap T = \{z\}$ ,  $A \cap D = \emptyset$ . Obviously,  $|X| \geq 3$ . Since  $|S \cap D| = 1$  and  $|D| \geq 2$ , we have that  $B \cap D \neq \emptyset$ , and so  $|X| \geq 4$ . However, by noticing that  $|A \cap T| = 1$ , we have that  $|(S \cup B) \cap T| = 2$ , and so  $|X| = 3$ , a contradiction. If  $bx \in E_N(G)$  or  $cx \in E_N(G)$ , we may employ a similar argument. So, next we may assume that  $bx, cx \in E_R(G)$ .
- (2) If  $xz \in E_N(G)$ , we take the corresponding separating group  $(xz, T; C, D)$  such that  $x \in C, z \in D$ . Then, we have that  $x \in A \cap C, z \in A \cap D$ . It is easy to see that  $a, b, c \in S \cap T$ . Since  $|T| = 3$ , we have that  $y \in B \cap C$ . Let  $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$ , and so  $|X| = 3$ . Then, we have that  $B \cap D = \emptyset$ . Noticing that  $D \cap S = \emptyset$ , we have that  $D = A \cap D = \{z\}$ , which contradicts to that  $|D| \geq 2$ . Therefore,  $xz \in E_R(G)$ .

From the above arguments, we know that the lemma holds.  $\square$

Now we present our main results. For convenience we denote by  $\mathfrak{R}$  the set of all helms, maximal  $l$ -bi-fans, maximal  $l$ -belts, maximal  $l$ -co-belts,  $W$ -frameworks and  $W'$ -frameworks of a graph  $G$ .

**Definition 4.4.** Let  $C$  be a cycle of a 4-connected graph  $G$  and  $H$  a subgraph of  $G$  belonging to  $\mathfrak{R}$ . If  $C$  contains an inner vertex of  $H$ , then we say that  $C$  passes through  $H$ .

**Theorem 4.5.** *Let  $G$  be a 4-connected graph and  $C$  a cycle of  $G$ . If  $C$  does not pass through any subgraph of  $G$  belonging to  $\mathfrak{R}$ , then there are least two removable edges of  $G$  in  $C$ .*

**Proof.** By contradiction. Assume that  $C$  does not pass through any subgraph of  $G$  belonging to  $\mathfrak{R}$ , and there is at most one removable edge of  $G$  in  $C$ . Let  $F = E(C) \cap E_R(G)$ , then  $|F| \leq 1$ . Denote  $E(C) - F$  by  $E_0$ . We take the separating group  $(uw, S'; A', B')$  such that  $u \in A', w \in B'$  and  $uw \in E_0$ . From  $|F| \leq 1$  we know that  $(E(A') \cup ([A', S']) \cap F = \emptyset$  or  $(E(B') \cup ([S', B']) \cap F = \emptyset$ . Without loss of generality, we may assume that  $(E(A') \cup ([A', S']) \cap F = \emptyset$ . Since  $A'$  is an  $E_0$ -edge-vertex-cut fragment,  $A'$  must contain an  $E_0$ -edge-vertex-cut end-fragment as its subgraph, say  $A$ . Then, we have that  $(E(A) \cup ([A, S]) \cap F = \emptyset$ , and we take a separating group  $(xy, S; A, B)$  such that  $x \in A, y \in B$  with  $xy \in E_0$ . Next, we will consider  $|A|$  by cases.

*Case 1:*  $|A| = 2$ . Then,  $A$  is a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom, say,  $A = \{x, z\}$ . Let  $S = \{a, b, c\}$ .

*Subcase 1.1:* If  $A$  is a 2-edge-vertex-cut atom, since  $xy \in E(C)$  and  $C$  is a cycle of  $G$ , we have that  $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$ . From Lemma 4.3 we know that  $\{xa, xb, xc, xz\} \subset E_R(G)$ , which contradicts to that  $(E(A) \cup [A, S]) \cap F = \emptyset$ .

*Subcase 1.2:* If  $A$  is a 1-edge-vertex-cut atom, by noticing that  $C$  is a cycle of  $G$  and  $([E(A) \cup [A, S]) \cap F = \emptyset$ , then obviously  $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$ . From Corollary 4.2 we know that  $x$  is an inner vertex of one of the subgraphs of  $G$  belonging to  $\mathfrak{R}$ . Since  $xy \in E(C)$ , this contradicts to that  $C$  does not pass through any subgraph of  $G$  belonging to  $\mathfrak{R}$ .

*Case 2:*  $|A| \geq 3$ . Then, we will discuss the following subcases.

*Subcase 2.1:* If there exists an  $xz \in E_0 \cap E(A \cup [A, S])$ , then obviously  $z \notin S$ ; otherwise, we would have  $|A| = 2$ , a contradiction to that  $|A| \geq 3$ . We take the separating group  $(xz, S_1; A_1, B_1)$  such that  $x \in A_1, z \in B_1$ . Then, we have that  $x \in A \cap A_1, z \in A \cap B_1$ . Let

$$X_1 = (A_1 \cap S) \cup (S \cap S_1) \cup (A \cap S_1),$$

$$X_2 = (A \cap S_1) \cup (S \cap S_1) \cup (B_1 \cap S),$$

$$X_3 = (B_1 \cap S) \cup (S \cap S_1) \cup (B \cap S_1),$$

$$X_4 = (B \cap S_1) \cup (S \cap S_1) \cup (A_1 \cap S).$$

If  $y \in B \cap S_1$ , from Theorem 2.2 we have that  $|A_1| = 2$ , say  $A_1 = \{x, v_1\}$ . We claim that  $A_1$  is a 1-edge-vertex-cut atom; otherwise,  $A_1$  is a 2-edge-vertex-cut atom, and then, from Lemma 4.3 we have  $xy \in E_R(G)$ , a contradiction. From Corollary 4.2 we know that  $x$  is an inner vertex of some subgraph of  $G$  belonging to  $\mathfrak{R}$ , a contradiction to the assumption. Therefore,  $y \notin B \cap S_1$ , and so  $y \in A_1 \cap B$ . Since  $A \cap B_1 \neq \emptyset$ , we have that  $X_2$  is a vertex-cut of  $G - xz$ , and so  $|X_2| \geq 3$ . By an analogous argument, we can deduce that  $|X_4| \geq 3$ . Since  $|X_2| + |X_4| = |S| + |S_1| = 6$ , we can get that  $|X_2| = |X_4| = 3$ , and so  $|A_1 \cap S| = |A \cap S_1|$ ,  $|B \cap S_1| = |B_1 \cap S|$ . We claim that  $A \cap B_1 = \{z\}$ . Otherwise,  $|A \cap B_1| \geq 2$ . Then,  $(xz, X_2; A \cap B_1, A_1 \cup B)$  is a separating group of  $G$  and  $xz \in E_0$ . It is easy to see that  $A \cap B_1$  is an  $E_0$ -edge-vertex-cut fragment contained in  $A$ , which contradicts to that  $A$  is an  $E_0$ -edge-vertex-cut end-fragment of  $G$ . Therefore,  $A \cap B_1 = \{z\}$ . Since  $|B_1| \geq 2$  and  $B_1$  is a connected subgraph of  $G$ , we have that  $B_1 \cap S \neq \emptyset$ .

*Subsubcase 2.1.1:* If  $|B_1 \cap S| = |B \cap S_1| = 3$ , then  $|X_1| = 0$ , and so  $\{z, y\}$  would be 2-vertex-cut of  $G$ , a contradiction.

*Subsubcase 2.1.2:* If  $|B_1 \cap S| = |B \cap S_1| = 2$ , since  $X_1$  is a vertex-cut of  $G - xy - xz$ , then  $|X_1| \geq 2$ . Noticing that  $|S| = |S_1| = 3$ , we have that  $|A \cap S_1| = |A_1 \cap S| = 1$ ,  $S \cap S_1 = \emptyset$ . We claim that  $A \cap A_1 = \{x\}$ . Otherwise,  $|A \cap A_1| \geq 2$ . Then,  $\{x\} \cup X_1$  would be a 3-vertex-cut of  $G$ , a contradiction. Let  $A \cap S_1 = \{a\}$ ,  $A_1 \cap S = \{b\}$ ,  $S \cap B_1 = \{v_1, v_2\}$ . From  $A \cap B_1 = \{z\}$  we can get that  $\Gamma_G(z) = \{x, a, v_1, v_2\}$ . We claim that  $ab \in E(G)$ . Otherwise,  $\{x, v_1, v_2\}$  would be a 3-vertex-cut of  $G$ , a contradiction. We claim that  $av_1, av_2 \in E(G)$ . Otherwise, without loss of generality, we may assume that  $av_1 \notin E(G)$ . Let  $A' = \{x, a\}$ ,  $S' = \{b, z, v_2\}$ ,  $B' = G - xy - S' - A'$ , then  $(xy, S'; A', B')$  is a separating group of  $G$ . Since  $xy \in E_0$ ,  $A'$  is an  $E_0$ -edge-vertex-cut fragment contained in  $A$ , which contradicts to that  $A$  is an  $E_0$ -edge-vertex-cut end-fragment. So,  $av_1, av_2 \in E(G)$ , and hence  $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$ . Let  $S_0 = \{x, v_1, v_2\}$ ,  $A_0 = \{a, z\}$ ,  $B_0 = G - ab - S_0 - A_0$ , then  $(ab, S_0; A_0, B_0)$  is a separating group of  $G$ , and so  $ab \in E_N(G)$ .

We claim that  $az \in E_R(G)$ . Otherwise,  $az \in E_N(G)$ , and we take the corresponding separating group  $(az, S'; A', B')$  such that  $a \in A'$ ,  $z \in B'$ . Since  $axza, av_1za, av_2za$  are 3-cycles of  $G$ , we have that  $x, v_1, v_2 \in S'$ . Since  $xz \in E_N(G)$ , from Theorem 2.2 we have that  $|B'| = 2$ , say  $B' = \{z, u\}$ . Then,  $uzxu$  is a 3-cycle of  $G$ , which is impossible to hold in  $G$ , and so  $az \in E_R(G)$ .

Since  $(E(A) \cup ([A, S])) \cap F = \emptyset$  and  $C$  is a cycle of  $G$ , we can get that  $\{zv_1, zv_2\} \cap E_N(G) \neq \emptyset$ . Without loss of generality, we may assume that  $zv_1 \in E_N(G)$ . We take the separating group  $(zv_1, T; C', D')$  such that  $z \in C'$ ,  $v_1 \in D'$ . Then, we have that  $z \in C' \cap B_1$ ,  $v_1 \in B_1 \cap D'$ . Obviously,  $a \in S_1 \cap T$ . Let

$$Y_1 = (A_1 \cap T) \cup (S_1 \cap T) \cup (C' \cap S_1),$$

$$Y_2 = (C' \cap S_1) \cup (S_1 \cap T) \cup (B_1 \cap T),$$

$$Y_3 = (B_1 \cap T) \cup (S_1 \cap T) \cup (S_1 \cap D'),$$

$$Y_4 = (D' \cap S_1) \cup (S_1 \cap T) \cup (A_1 \cap T).$$

- (1) If  $x \in A_1 \cap C'$ , then  $Y_1$  is a vertex-cut of  $G - xz$ , and so  $|Y_1| \geq 3$ . By a similar argument, we have that  $|Y_3| \geq 3$ . Since  $|Y_1| + |Y_3| = |S_1| + |T| = 6$ , we can conclude that  $|Y_1| = |Y_3| = 3$  and  $|A_1 \cap T| = |S_1 \cap D'|$ ,  $|S_1 \cap C'| = |B_1 \cap T|$ . Since  $a \in S_1$ , from Theorem 2.4 we know that  $b \notin T \cup S_1$ . Since  $bx, zv_2 \in E(G)$ , we have that  $b \in A_1 \cap C'$  and  $v_2 \notin D' \cap B_1$ . From  $\Gamma_G(a) = \{v_1, v_2, z, x, b\}$ , we know that  $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$ . Then, we have that  $|A_1 \cap T| = |S_1 \cap D'| = 0, 1$  or  $2$ .
  - (1.1) If  $|A_1 \cap T| = |D' \cap S_1| = 2$ , then  $|S_1 \cap C'| = |B_1 \cap T| = 0$ . Since  $zv_2 \in E(G)$ , we have  $v_2 \in B_1 \cap C'$ , and hence  $\{a, z\}$  would be 2-vertex-cut of  $G$ , a contradiction.
  - (1.2) If  $|A_1 \cap T| = |D' \cap S_1| = 1$ , then  $|S_1 \cap T| \leq 2$ . First, we claim that  $B_1 \cap D' = \{v_1\}$ . Otherwise,  $|B_1 \cap D'| \geq 2$ . Then, from  $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$ , we can conclude that  $\{v_1\} \cup (Y_3 - \{a\})$  would be a 3-vertex-cut of  $G$ , a contradiction. So,  $B_1 \cap D' = \{v_1\}$ . Let  $D' \cap S_1 = \{u_1\}$ . If  $A_1 \cap D' \neq \emptyset$ , from  $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$  we can get that  $A_1 \cap D' \cap \Gamma_G(a) = \emptyset$ , and so  $Y_4 - \{a\}$  would be a vertex-cut of  $G$  with cardinality less than 4, a contradiction. Therefore,  $A_1 \cap D' = \emptyset$ . Then,  $au_1 \in E(G)$ . However, it is easy to see that  $u_1 \notin \{x, z, b, v_1, v_2\}$ , a contradiction.
  - (1.3) If  $|D' \cap S_1| = |A_1 \cap T| = 0$ , since  $D'$  is a connected subgraph of  $G$ , we have that  $A_1 \cap D' = \emptyset$ . Then,  $|D'| = |D' \cap B_1| \geq 2$ . Since  $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$ , by noticing that  $|Y_3| = 3$ , we have that  $\{v_1\} \cup (Y_3 - \{a\})$  would be a 3-vertex-cut of  $G$ , a contradiction.
- (2) If  $x \in A_1 \cap T$ , from Theorem 2.2 we have that  $|C'| = 2$ . Since  $C'$  is a connected subgraph of  $G$ , we have that  $A_1 \cap C' = \emptyset$ . If  $S_1 \cap C' \neq \emptyset$ , since  $a \in S_1 \cap T$ , then  $|D' \cap S_1| \leq 1$ . Noticing that  $Y_3$  is a vertex-cut of  $G - zv_1$ , we have that  $|Y_3| \geq 3$ , and so  $|B_1 \cap T| = 1$ ,  $A_1 \cap T = \{x\}$ . Obviously,  $|Y_4| = 3$ , and hence  $A_1 \cap D' = \emptyset$ , and so  $A_1 = \{x\}$ , which contradicts to that  $|A_1| \geq 2$ . So, we have that  $S_1 \cap C' = \emptyset$ , and so  $|B_1 \cap C'| = 2$ . Since  $A_1 \cap T \neq \emptyset$ , obviously,  $\{z\} \cup (T - \{x\})$  would be a vertex-cut with cardinality less than 4, a contradiction.

From the above arguments, we can conclude that subsubcase 2.1.2 does not occur.

*Subsubcase 2.1.3:* If  $|B_1 \cap S| = |B \cap S_1| = 1$ , then  $|S \cap S_1| \leq 2$ . We claim that  $|S \cap S_1| < 2$ . Otherwise,  $|S \cap S_1| = 2$ . Then,  $A \cap S_1 = \emptyset = S \cap A_1$ . If  $|A \cap A'| \geq 2$ , then  $\{x\} \cup (S \cap S_1)$  would be a vertex-cut of  $G$  with cardinality less than 4, a contradiction, and so  $A \cap A_1 = \{x\}$ . Note that  $|X_2| = 3$ . If  $|A \cap B_1| \geq 2$ , then by an argument similar to that used in Subcase 2.1,  $A \cap B_1$  would be an  $E_0$ -edge-vertex-cut fragment contained in  $A$ , which contradicts to that  $A$  is an  $E_0$ -edge-vertex-cut end-fragment. Hence,  $A \cap B_1 = \{z\}$ , and so  $|A| = 2$ , which contradicts to that  $|A| \geq 3$ . Therefore,  $|S \cap S_1| \leq 1$ , and then  $|X_3| \leq 3$ , and so  $B \cap B_1 = \emptyset$ . Since  $A \cap B_1 = \{z\}$ , we have that  $|B_1| = 2$  and  $B_1$  is a 1-edge-vertex-cut atom of  $G$ , say  $B_1 = \{z, u\}$ . Since  $C$  is a cycle and  $(E(A) \cup \{A, S\}) \neq \emptyset$ , we have that  $z$  is incident with at least two unremovable edges. From Corollary 4.2 we know that  $z$  is an inner vertex of some subgraph of  $G$  belong to  $\mathfrak{R}$ , which contradicts to that  $C$  does not pass through any subgraph of  $G$  belonging to  $\mathfrak{R}$ . The proof is now complete.  $\square$

**Theorem 4.6.** *Let  $G$  be a 4-connected graph and  $C$  a cycle of  $G$ . If  $C$  passes through only one subgraph of  $G$  belonging to  $\mathfrak{R}$ , then there exists at least one removable edge of  $G$  in  $C$ .*

**Proof.** By contradiction. Assume that  $E(C) \subset E_N(G)$ . Let  $C$  pass through the subgraph  $H$  of  $G$  that belongs to  $\mathfrak{R}$ ; see the definitions of  $H$  in Definitions 3.1 through 3.6. If  $H$  is a maximal  $l$ -belt, from the assumption, it is easy to see that  $\{x_2x_1, y_{l+1}y_{l+2}\} \cap E(C) \neq \emptyset$ . If  $x_2x_1 \in E(C)$ , by letting  $S = \{y_{l+2}, x_{l+2}, y_1\}$ ,  $e = x_2x_1$ ,  $B = \{x_2, \dots, x_{l+1}, y_2, \dots, y_{l+1}\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the maximal  $l$ -belt ( $l \geq 1$ ); if  $y_{l+1}y_{l+2} \in E(C)$ , by letting  $S = \{x_1, y_1, x_{l+2}\}$ ,  $e = y_{l+1}y_{l+2}$ ,  $B = \{x_2, \dots, x_{l+1}, y_2, \dots, y_{l+1}\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the maximal  $l$ -belt ( $l \geq 1$ ). If  $H$  is a maximal  $l$ -co-belt, similarly, we have that  $\{x_1x_2, y_1y_2\} \cap E(C) \neq \emptyset$ , if  $x_1x_2 \in E(C)$ , by letting  $S = \{y_{l+2}, x_{l+3}, y_1\}$ ,  $e = x_2x_1$ ,  $B = \{x_2, \dots, x_{l+2}, y_2, \dots, y_{l+1}\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the maximal  $l$ -co-belt ( $l \geq 1$ ); if  $y_1y_2 \in E(C)$ , by letting  $S = \{y_{l+2}, x_{l+3}, x_2\}$ ,  $e = y_2y_1$ ,  $B = \{x_3, \dots, x_{l+2}, y_2, \dots, y_{l+1}\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the maximal  $l$ -co-belt ( $l \geq 1$ ). If  $H$  is a maximal  $l$ -bi-fan ( $l \geq 1$ ), by letting  $S = \{a, b, x_{l+3}\}$ ,  $e = x_2x_1$ ,  $B = \{x_2, \dots, x_{l+2}\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the maximal  $l$ -bi-fan. If  $H$  is a helm, by letting  $e = x_1v_1$ ,  $S = \{v_2, v_3, v_4\}$ ,  $B = \{a, x_1, x_2, x_3, x_4\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the helm. If  $H$  is a  $W$ -framework, then  $C$  must pass through  $x_1x_2, x_2x_3$ . In this case, by letting  $e = x_2x_1$ ,  $S = \{x_3, y_4, y_2\}$ ,  $B = \{x_2, y_3\}$ ,  $A = G - e - S - B$ , then  $(e, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the  $W$ -framework. If  $H$  is a  $W'$ -framework, by noticing that  $\{x_1x_2, x_2y_2\} \cap E(C) \neq \emptyset$ , then if  $x_1x_2 \in E(C)$ , by letting  $S = \{y_2, x_3, y_4\}$ ,  $B = \{x_2, y_3\}$ ,  $A = G - x_1x_2 - S - B$ , then  $(x_1x_2, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the  $W'$ -framework; if  $x_2y_2 \in E(C)$ , by letting  $S = \{x_1, y_3, v\}$  such that  $v \in \Gamma_G(x_3)$ ,  $B = \{x_2, x_3\}$ ,  $A = G - x_2y_2 - S - B$ , then the separating group  $(x_2y_2, S; A, B)$  is a separating group of  $G$  such that  $A$  does not contain any inner vertex of the  $W'$ -framework.

Let  $E_0 = E(C)$ , then  $A$  is an  $E_0$ -edge-vertex-cut fragment of  $G$  such that it does not contain any inner vertex of  $H$ . Obviously,  $A$  contains an  $E_0$ -edge-vertex-cut end-fragment as its subgraph, say  $A'$ . It is easy to see that  $A'$  does not contain any inner vertex of  $H$ . Finally, by an argument analogous to that used in the proof of Theorem 4.5, we can show that  $A'$  contains an inner vertex of some subgraph of  $G$  belonging to  $\mathfrak{R}$ , which contradicts to that  $A'$  does not contain any inner vertex of any subgraph of  $G$  belonging to  $\mathfrak{R}$ . The proof is now complete.  $\square$

Finally, to end this paper we construct examples to show that the lower bounds for the numbers of removable edges of  $G$  that a cycle of  $G$  can contain in Theorems 4.5 and 4.6 are in some sense best possible, and we also construct an example to show that the conditions, i.e., the numbers of subgraphs of  $G$  belonging to  $\mathfrak{R}$  that a cycle of  $G$  can pass through in Theorems 4.5 and 4.6 are in some sense best possible.

Let  $F$  be a maximal  $k$ -bi-fan such that  $V(F) = \{a, b, z_1, z_2, \dots, z_{k+3}\}$  and  $E(F) = \{z_1z_2, z_2z_3, \dots, z_{k+2}z_{k+3}, az_2, az_3, \dots, az_{k+2}, bz_2, \dots, bz_{k+2}\}$  where  $k \geq 1$ . Let  $L$  be a maximal  $l$ -belt such that  $V(L) = \{x_1, x_2, \dots, x_{l+2}, y_1, y_2, \dots, y_{l+2}\}$  and  $E(H) = E_1(H) \cup E_2(H)$ , where  $E_1(H) = \{x_1x_2, x_2x_3, \dots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \dots, y_{l+1}y_{l+2}\}$  and  $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \dots, y_{l+1}x_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$ , in which  $l \geq 1$ .

**Example 1.** Identify the vertex  $a$  with  $x_1$ , vertex  $b$  with  $y_{l+2}$ , vertex  $z_{k+3}$  with  $x_{l+2}$ , vertex  $z_1$  with  $y_1$ , respectively. Denote the resulting graph by  $G_1$ . Let  $G = G_1 + ab + y_1x_{l+2}$ . It is easy to see that  $G$  is a 4-connected graph. First, let  $A = \{x_3, x_4, \dots, x_{l+1}, y_2, y_3, \dots, y_{l+1}\}$ ,  $S = \{x_2, x_{l+2}, y_1\}$ ,  $B = G - by_{l+1} - S - A$ , then  $(by_{l+1}, S; A, B)$  is a separating group of  $G$ , and so  $by_{l+1} \in E_N(G)$ . Since  $y_1x_{l+2} \in E([S])$ , from Theorem 2.4 we have that  $y_1x_{l+2} \in E_R(G)$ . Obviously,  $(x_{l+2}z_{k+2}, S_1)$  is a separating pair such that  $S_1 = \{a, b, z_2\}$ , and  $(z_2y_1, S_2)$  is also a separating pair such that  $S_2 = \{a, b, x_{l+2}\}$ . It is easy to see that  $z_i z_{i+1} \in E_N(G)$ , where  $i = 2, \dots, k+1$ . Pick up the cycle  $C_1 = y_1x_{l+2}z_{k+2}z_{k+1}z_k \cdots z_2y_1$ . Then,  $C_1$  only passes through one subgraph of  $G$  belonging to  $\mathfrak{R}$ , and  $C_1$  has only one removable edge  $y_1x_{l+2}$  of  $G$ . This shows that the result of Theorem 4.6 is in some sense best possible.



**Example 2.** First, delete the vertices  $z_1, z_{k+3}$  from  $F$ . Then, identify vertex  $z_2$  with  $x_1$ , vertex  $z_{k+2}$  with  $y_{l+2}$ , respectively. Denote the resulting graph by  $G_2$ . Let  $G = G_2 + ab + ay_1 + bx_{l+2} + y_1x_{l+2}$ . It is easy to see that  $G$  is a 4-connected graph. Let  $A = \{x_3, \dots, x_{l+1}, y_2, \dots, y_{l+1}\}$ ,  $S = \{y_1, x_{l+2}, x_2\}$ ,  $B = G - z_{k+2}y_{l+1} - S - A$ , then  $(z_{k+2}y_{l+1}, S; A, B)$  is a separating group of  $G$ , and so  $z_{k+2}y_{l+1} \in E_N(G)$ . Since  $y_1x_{l+2} \in E([S])$ , from Theorem 2.4 we have that  $y_1x_{l+2} \in E_R(G)$ . Obviously,  $(z_2x_2, S_1)$  is a separating group of  $G$  such that  $S_1 = \{a, b, z_{k+2}\}$ , and so  $z_2x_2 \in E_N(G)$ . By a similar argument, we can get that  $ay_1, bx_{l+2} \in E_N(G)$ . Since  $ab \in E([S_1])$ , we have  $ab \in E_R(G)$ . Pick up the cycle  $C_2 = abx_{l+2}y_1a$ . Then,  $C_2$  does not pass through any subgraph of  $G$  belonging to  $\mathfrak{R}$ , and  $C_2$  has exactly two removable edges  $ab, y_1x_{l+2}$  of  $G$ . This shows that the result of Theorem 4.5 is in some sense best possible.

The following example shows that if a cycle  $C$  of  $G$  passes through two subgraphs of  $G$  belonging to  $\mathfrak{R}$ , then it may not contain any removable edge of  $G$ .

**Example 3.** First, delete the vertices  $z_{k+3}$  from  $F$ . Then, identify the vertex  $a$  with  $x_1$ , vertex  $b$  with  $x_{l+2}$ , vertex  $z_{k+2}$  with  $y_{l+2}$ , vertex  $z_1$  with  $y_1$ , respectively. Denote the resulting graph by  $G_3$ . Let  $G = G_3 + ab + y_1x_{l+2}$ . It is easy to see that  $G$  is a 4-connected graph. Pick up the cycle  $C_3 = y_1y_2 \cdots y_{l+2}z_{l+2}z_{l+1} \cdots z_2y_1$ . Then,  $C_3$  passes through two subgraphs of  $G$  belonging to  $\mathfrak{R}$ . It is easy to see that  $E(C_3) \subset E_N(G)$ , and so  $C_3$  does not contain any removable edge of  $G$ . This in some sense shows that the conditions of Theorems 4.5 and 4.6 are best possible.

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