Optimal Embeddings of Paths with Various Lengths in Twisted Cubes

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Abstract—Twisted cubes are variants of hypercubes. In this paper, we study the optimal embeddings of paths of all possible lengths between two arbitrary distinct nodes in twisted cubes. We use TQ_n to denote the $n\text{-dimensional}$ twisted cube and use $\text{dist}(TQ_n, u, v)$ to denote the distance between two nodes u and v in TQ_n, where $n \geq 1$ is an odd integer. The original contributions of this paper are as follows: 1) We prove that a path of length l can be embedded between u and v with dilation 1 for any two distinct nodes u and v and any integer l with $\text{dist}(TQ_n, u, v) + 2 \leq l \leq 2^n - 1$ $(n \geq 3)$ and 2) we find that there exist two nodes u and v such that no path of length $\text{dist}(TQ_n, u, v) + 1$ can be embedded between u and v with dilation 1 $(n \geq 3)$. The special cases for the nonexistence and existence of embeddings of paths between nodes u and v and with length $\text{dist}(TQ_n, u, v) + 1$ are also discussed. The embeddings discussed in this paper are optimal in the sense that they have dilation 1.

Index Terms—Twisted cube, interconnection network, path, edge-pancyclicity, embedding, dilation.

1 Introduction

Interconnection networks take a key role in parallel computing systems. An interconnection network can be represented by a graph $G = (V, E)$, where V represents the node set and E represents the edge set. In this paper, we use graphs and interconnection networks interchangeably.

Graph embedding is to embed a graph into another graph. It can be formally defined as: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an embedding from G_1 to G_2 is an injective mapping ψ: $V_1 \rightarrow V_2$. G_1 and G_2 are called *guest* graph and *host* graph, respectively. An important performance metric of embedding is *dilation*. The dilation of embedding ψ is defined as

$$\text{dil}(G_1, G_2, \psi) = \max\{\text{dist}(G_2, \psi(u), \psi(v)) | (u, v) \in E_1\},$$

where $\text{dist}(G_2, \psi(u), \psi(v))$ denotes the distance between the two nodes $\psi(u)$ and $\psi(v)$ in G_2. The smaller the dilation of an embedding is, the shorter the communication delay that the graph G_2 simulates the graph G_1. We call ψ the optimal embedding from G_1 to G_2 if ψ has the smallest dilation in all the embeddings from G_1 to G_2. Clearly, the dilation of the optimal embedding is at least 1. Under this circumstance, G_1 is a subgraph of G_2. Finding the optimal embedding of graphs is NP-hard.

Many graph embeddings take cycles, trees, meshes, paths, etc., as guest graphs [3], [10], [12], [14], [16], [21], [22] because these interconnection networks are widely used in parallel computing systems. Path embeddings are especially important because paths are the common structures used to model linear arrays in parallel processing [5], [6], [17], [18], [19], [20].

Twisted cubes [1], [13] are variants of hypercubes. The $n\text{-dimensional}$ twisted cube has 2^n nodes and $n2^{n-1}$ edges. It possesses some desirable features for interconnection networks [13]. Its diameter, wide diameter, and faulty diameter are about half of those of the $n\text{-dimensional}$ hypercube [4]. A complete binary tree can be embedded into it [2]. It has the same diagnosability as the $n\text{-dimensional}$ hypercube under the $t/k\text{-diagnosis}$ strategy based on the well-known PMC diagnostic model [9]. It was shown that it is pancyclic [4] and $(n-2)\text{-Hamiltonian}$ and $(n-3)\text{-Hamiltonian}$ connected [15]. Recently, it was proven that it has edge-pancyclicity [11], which is a stronger property compared with its pancyclicity.

In this paper, we discuss the optimal embeddings of paths of various lengths between any two nodes in twisted cubes. We use TQ_n to denote the $n\text{-dimensional}$ twisted cube and use $\text{dist}(TQ_n, u, v)$ to denote the distance between two nodes u and v in TQ_n, where $n \geq 1$ is an odd integer. The original contributions of this paper are as follows:

1. We prove that a path of length l can be embedded between u and v with dilation 1 for any two distinct nodes u and v and any integer l with $\text{dist}(TQ_n, u, v) + 2 \leq l \leq 2^n - 1$ $(n \geq 3)$.
2. We find that there exist two nodes u and v such that no path of length $\text{dist}(TQ_n, u, v) + 1$ can be embedded between u and v with dilation 1 $(n \geq 3)$.

The special cases for the nonexistence and existence of embeddings of paths between nodes u and v and with length $\text{dist}(TQ_n, u, v) + 1$ are also discussed.

The embeddings discussed in this paper are optimal in the sense that they have dilation 1.

The rest of this paper is organized as follows: In Section 2, we give some definitions and notations used in

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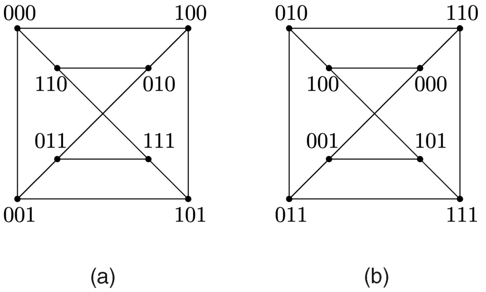


Fig. 1. The three-dimensional twisted cube TQ_3, where (a) and (b) demonstrate two different drawings of TQ_3.

the paper. In Section 3, we study the embeddings of paths of all possible lengths between arbitrary two distinct nodes with dilation 1 in twisted cubes. Section 3 discusses the nonexistence and existence of embeddings of paths of length $\mathrm{dist}(TQ_n, u, v) + 1$ for two nodes u and v in TQ_n. The final section concludes this paper.

2 Preliminaries

Let $G = (V, E)$ be a graph. A path P from node u to node v in G is denoted by P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$. Nodes u and v are called the two *end nodes* of path P. If $u = v$, then P is called a *cycle*. Path P can also be denoted by

$$u = u^{(0)}, u^{(1)}, \ldots, u^{(i-1)}, P_1, u^{(j+1)}, u^{(j+2)}, \ldots, u^{(k)} = v,$$

where P_1 is the subpath of P from $u^{(i)}$ to $u^{(j)}$, i.e., $u^{(i)}, u^{(i+1)}, \ldots, u^{(j)}$ $(i \le j)$. The subpath P_1 can be denoted by $\mathrm{path}(P, u^{(i)}, u^{(j)})$. The length of path P is denoted by $\mathrm{len}(P)$. The node set of P is denoted by $V(P)$.

Let C: $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v, u$ be a cycle in G. We use $C - (u, v)$ to denote the path $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$ after deleting the edge (u, v) in C and use $C - (v, u)$ to denote the path $v = u^{(k)}, u^{(k-1)}, \ldots, u^{(0)} = u$ after deleting the edge (v, u) in C.

For $u, v \in V(G)$, we call v to be a neighbor of u if $(u, v) \in E(G)$. The distance between u and v is defined as $\mathrm{dist}(G, u, v) = \min\{\mathrm{len}(P) | P \text{ is a path between } u \text{ and } v \text{ in } G\}$. The diameter of G is defined as

$$\mathrm{diam}(G) = \max\{\mathrm{dist}(G, u, v) | u, v \in V(G)\}.$$

G is called a *pancyclic* graph if G contains any cycle of length l with $3 \le l \le |V|$, i.e., any cycle of length l with $3 \le l \le |V|$ can be embedded into G with dilation 1. However, there is no cycle of length 3 in twisted cubes. For convenience of discussion in this paper, we call G a *pancyclic* graph if G contains any cycle of length l with $4 \le l \le |V|$. Similarly, we define *edge-pancyclic* graphs as follows.

G is called an *edge-pancyclic* graph if, for every edge (u, v) and any integer l with $4 \le l \le |V|$, any cycle of length l can be embedded into G with dilation 1 such that (u, v) is in this cycle.

Given two graphs $G' = (V', E')$ and $G'' = (V'', E'')$, if there exists a bijection φ from V' to V'' such that $(u', v') \in E'$ if and only if $(\varphi(u'), \varphi(v')) \in E''$ for any two nodes $u', v' \in V'$, then we say that G' is *isomorphic* to G''.

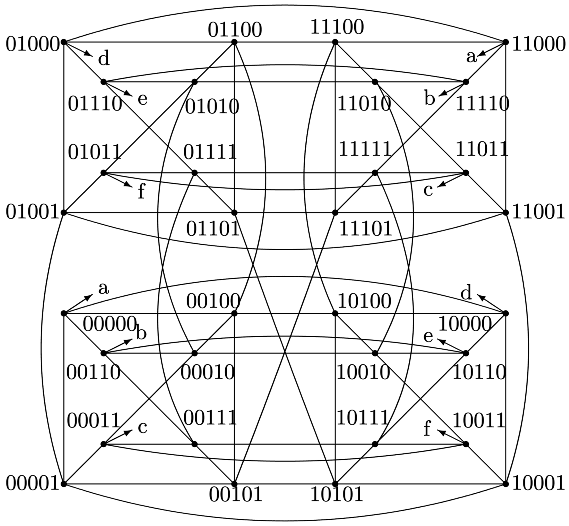


Fig. 2. The five-dimensional twisted cube TQ_5, where the end nodes of a missing edge are marked with arrows labeled with the same letter.

Let G_1 and G_2 be two subgraphs of G. We use $G_1 \bigcup G_2$ to denote the subgraph induced by the node subset $V(G_1) \bigcup V(G_2)$ in G. The *Cartesian product* of G_1 and G_2 is defined as the graph $G_1 \times G_2$, where $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and, for any $x, y \in V(G_1 \times G_2)$ with $x = (u_1, u_2)$ and $y = (v_1, v_2)$, $(x, y) \in E(G_1 \times G_2)$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

A binary string u of length n is denoted by $u_{n-1}u_{n-2}\ldots u_0$. The ith bit u_i of u can also be written as $\mathrm{bit}(u, i)$. The complement of u_i is denoted by $\overline{u_i} = 1 - u_i$. In [13], the n-dimensional twisted cube TQ_n was defined. It is an n-regular graph with 2^n nodes and $n2^{n-1}$ edges, where n is an odd integer. We label all the nodes of TQ_n by binary strings of length n. In this paper, we do not distinguish between the nodes of TQ_n and their labels. If $u = u_{n-1}u_{n-2}\ldots u_0 \in V(TQ_n)$, for $0 \le i \le n-1$, we define $f(u, i) = u_i \bigoplus u_{i-1} \bigoplus \ldots \bigoplus u_0$, where $\bigoplus$ is the exclusive operation. According to the definition of TQ_n in [13], we may give a recursive definition of TQ_n for any odd integer $n \ge 1$ as follows:

Definition 1. *The one-dimensional twisted cube TQ_1 is defined as the complete graph with two nodes labeled 0 and 1. For an odd integer $n \ge 3$, TQ_n consists of four subcubes, TQ^{00}_{n-2}, TQ^{01}_{n-2}, TQ^{10}_{n-2}, and TQ^{11}_{n-2}, where TQ^{ab}_{n-2} is isomorphic to TQ_{n-2} and $V(TQ^{ab}_{n-2}) = \{abx | x \in V(TQ_{n-2})\}$ and $E(TQ^{ab}_{n-2}) = \{(abx, aby) | (x, y) \in E(TQ_{n-2})\}$ for any $a, b \in \{0, 1\}$ and*

$$V(TQ_n) = \bigcup_{a,b\in\{0,1\}} V(TQ^{ab}_{n-2}), E(TQ_n) = \bigcup_{a,b\in\{0,1\}} E(TQ^{ab}_{n-2}) \bigcup E',$$

where, for the nodes $u = u_{n-1}u_{n-2}\ldots u_0$,

$$v = v_{n-1}v_{n-2}\ldots v_0 \in V(TQ_n),$$

$(u, v) \in E'$ if u and v satisfy one of the following conditions:

1. $u = \overline{v_{n-1}}v_{n-2}v_{n-3}\ldots v_0$,
2. $u = \overline{v_{n-1}}\ \overline{v_{n-2}}v_{n-3}\ldots v_0$ *and* $f(u, n-3) = 0$, *or*
3. $u = v_{n-1}\ \overline{v_{n-2}}v_{n-3}\ldots v_0$ *and* $f(u, n-3) = 1$.

Fig. 1 and Fig. 2 show TQ_3 and TQ_5, respectively.

Notation 1. *For $n \geq 3$ and $a, b \in \{0,1\}$, the four (n − 2)-dimensional subcubes of TQ_n are denoted by TQ^{ab}_{n-2}, $TQ^{(1-a)b}_{n-2}$, $TQ^{a(1-b)}_{n-2}$, and $TQ^{(1-a)(1-b)}_{n-2}$, respectively. For example, if $a = 0$ and $b = 1$, then TQ^{ab}_{n-2} denotes TQ^{01}_{n-2} and $TQ^{(1-a)(1-b)}_{n-2}$ denotes TQ^{10}_{n-2}.*

For convenience, in the following sections, the parameter n always denotes an odd integer.

3 Embedding Paths of Various Lengths between Any Two Distinct Nodes

In this section, we study the optimal embeddings of paths of all possible lengths. Theorem 2 is the major result, which states that a path with length l can be embedded between any two nodes u and v with dilation 1 in TQ_n, where $\mathrm{dist}(TQ_n, u, v) + 2 \leq l \leq 2^n - 1$ ($n \geq 3$). To prove Theorem 2, we need to introduce Lemmas 1, 2, 3, and 4 and Theorem 1. Lemma 1 discusses the existence of a shortest path between any two nodes in TQ_n and the special properties of this shortest path. The other three lemmas and Theorem 1 are from [7], [10], [11], and [13], respectively.

To prove Lemma 1, we need to use the shortest path routing algorithm proposed by Abraham and Padmanabhan in [1]. The 0th "double bit" of node $u = u_{n-1}u_{n-2}\ldots u_0$ is defined as the single bit u_0 and the jth "double bit" is defined as $u_{2j}u_{2j-1}$ for $1 \leq j \leq \frac{n-1}{2}$. Let u, v be any two nodes in TQ_n. The double Hamming distance of u and v, denoted by $h_d(u, v)$, is defined as the number of different double bits between u and v. Clearly, $\mathrm{dist}(TQ_n, u, v) \geq h_d(u, v)$.

By using the algorithm proposed in [1], a shortest path between any two nodes can be found. Let u and v be two nodes in TQ_n. Let $z = u$. The detailed algorithm is described as follows:

1. If $z = v$, then the path is determined.
2. Assume that there exist neighbors w of z such that $h_d(w, v) = h_d(z, v) - 1$. Let w' be the w that differs from z with the largest double bit. Then, reset z to be w'.
3. Assume that all the neighbors of z, say w, satisfy $h_d(w, v) \geq h_d(z, v)$. Let j be the smallest index of double bits that z differs from v. Choose w' to be the neighbor of z that differs from z in the $2j$th bit. Then, reset z to be w'.

We call this shortest path routing algorithm the *AP algorithm*. By this algorithm, we will prove the following lemma:

Lemma 1. *For any $u, v \in V(TQ_n)$ and any $a, b \in \{0,1\}$, we have:*

1. *If $u, v \in V(TQ^{ab}_{n-2})$, then there exists a shortest path P between u and v in TQ_n such that P is in TQ^{ab}_{n-2}.*
2. *If $u \in V(TQ^{ab}_{n-2})$ and $v \in V(TQ^{(1-a)b}_{n-2})$, then there exists a shortest path P: $v^{(0)} = u, v^{(1)}, \ldots, v^{(l)} = v$ between u and v in TQ_n such that*
$$\{v^{(0)}, v^{(1)}, \ldots, v^{(k)}\} \subset V(TQ^{ab}_{n-2})$$
and $\{v^{(k+1)}, v^{(k+2)}, \ldots, v^{(l)}\} \subset V(TQ^{(1-a)b}_{n-2})$ for a certain integer k with $0 \leq k \leq l-1$.
3. *If $u \in V(TQ^{ab}_{n-2} \bigcup TQ^{(1-a)b}_{n-2})$ and*
$$v \in V\Big(TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2}\Big),$$
then there exists a shortest path P: $v^{(0)} = u, v^{(1)}, \ldots, v^{(l)} = v$ between u and v in TQ_n such that $\{v^{(0)}, v^{(1)}, \ldots, v^{(k)}\} \subset V(TQ^{ab}_{n-2} \bigcup TQ^{(1-a)b}_{n-2})$ and $\{v^{(k+1)}, v^{(k+2)}, \ldots, v^{(l)}\} \subset V(TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2})$ for a certain integer k with $0 \leq k \leq l-1$.

Proof.

1. Let P: $v^{(0)} = u, v^{(1)}, \ldots, v^{(l)} = v$ be a shortest path between u and v in TQ_n achieved by using the AP algorithm. Then, for any $v^{(k)} \in P$ with $1 \leq k \leq l$, we always have
$$\mathrm{bit}(v^{(k)}, n-1)\mathrm{bit}(v^{(k)}, n-2) = \mathrm{bit}(v^{(0)}, n)\mathrm{bit}(v^{(0)}, n-1) = ab.$$
Thus, we are done.
2. Let $v^{(0)} = u_{n-1}u_{n-2}\ldots u_0$, $v^{(1)} = \overline{u_{n-1}}u_{n-2}\ldots u_0$. Then, $v^{(1)} \in V(TQ^{(1-a)b}_{n-2})$, $u_{n-1}u_{n-2} = ab$, and $h_d(v^{(1)}, v) = h_d(v^{(0)}, v) - 1$. By the AP algorithm, $v^{(1)}$ is in a shortest path between u and v in TQ_n. By item 1, we can let P_1 be a shortest path between $v^{(1)}$ and v in TQ_n such that P_1 is in $TQ^{(1-a)b}_{n-2}$. Then,
$$v^{(0)} = u, P_1$$
is a shortest path between u and v in TQ_n, where $\{u\} \subset V(TQ^{ab}_{n-2})$ and $V(P_1) \subset V(TQ^{(1-a)b}_{n-2})$. Thus, we are done.
3. Let P: $u^{(0)} = u, u^{(1)}, \ldots, u^{(l)} = v$ be a shortest path between u and v in TQ_n. Since $u^{(0)} = u \in V(TQ^{ab}_{n-2} \bigcup TQ^{(1-a)b}_{n-2})$ and
$$u^{(l)} = v \in V\Big(TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2}\Big),$$
there must exist an integer m such that $v^{(m)} \in V(P) \bigcap V(TQ^{ab}_{n-2} \bigcup TQ^{(1-a)b}_{n-2})$ and
$$v^{(m+1)} \in V(P) \bigcap V\Big(TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2}\Big).$$
Let k be the smallest integer such that $v^{(k)} \in V(TQ^{ab}_{n-2} \bigcup TQ^{(1-a)b}_{n-2})$ and
$$v^{(k+1)} \in V\Big(TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2}\Big).$$
By items 1 and 2, we can let P_1 be a shortest path between $v^{(k+1)}$ and v in TQ_n such that P_1 is in $TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2}$. Then,
$$u^{(0)} = u, u^{(1)}, \ldots, u^{(k)}, P_1$$
is a shortest path between u and v in TQ_n, where $\{u^{(0)}, u^{(1)}, \ldots, u^{(k)}\} \subset V(TQ^{ab}_{n-2} \bigcup TQ^{(1-a)b}_{n-2})$ and $V(P_1) \subset V(TQ^{a(1-b)}_{n-2} \bigcup TQ^{(1-a)(1-b)}_{n-2})$. Thus, we are done. □

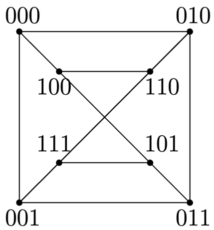


Fig. 3. The three-dimensional crossed cube CQ_3.

Crossed cubes are also variants of hypercubes [7]. The n-dimensional crossed cube is denoted by CQ_n, which has the same degree, node number, and edge number as TQ_n. CQ_3 is shown in Fig. 3. In [11], the edge-pancyclicity of crossed cubes was proven:

Theorem 1 [11]. *TQ_n is an edge-pancyclic graph for $n \geq 3$.*

Obviously, Theorem 1 is equivalent to the following corollary:

Corollary 1. *For any integer $n \geq 3$, any $x, y \in V(TQ_n)$ with $\mathrm{dist}(TQ_n, x, y) = 1$, and any integer l with $3 \leq l \leq 2^n - 1$, a path of length l can be embedded between x and y with dilation 1 in TQ_n.*

On the other hand, we introduce the following lemma in [11]:

Lemma 2 [11]. *TQ_3 is isomorphic to CQ_3.*

The following lemma gives a result on path embedding in CQ_n [10]:

Lemma 3 [10]. *If $n \geq 3$, for any $u, v \in V(CQ_n)$, $u \neq v$, and any integer l, $\lceil \frac{n+1}{2} \rceil + 1 \leq l \leq 2^n - 1$, there exists a path of length l between u and v in CQ_n.*

Lemma 4 provides the diameter of TQ_n [13]:

Lemma 4 [13]. $\mathrm{diam}(TQ_n) = \lceil \frac{n+1}{2} \rceil$.

With the above lemmas and corollary, we will prove the major result on path embedding in TQ_n in the following theorem. We will adopt induction on n to prove this theorem. In the induction part of the proof, we identify three cases according to the locations of two specific nodes. In each case, we further deal with many subcases according to the size of the given length of embedded path. This makes the proof long.

Theorem 2. *For any integer $n \geq 3$, any $u, v \in V(TQ_n)$ with $u \neq v$, and any integer l with $\mathrm{dist}(TQ_n, u, v) + 2 \leq l \leq 2^n - 1$, a path of length l can be embedded between u and v with dilation 1 in TQ_n.*

Proof. We use induction on n.

By Lemma 4, for any $u, v \in V(TQ_n)$ with $u \neq v$, we have $\mathrm{dist}(TQ_n, u, v) \in \{1, 2\}$. By Lemma 2 and Lemma 3, we can easily verify that the theorem holds when $n = 3$.

Supposing that the theorem holds for $n = \tau - 2$ ($\tau \geq 5$), we consider the case for $n = \tau$.

Let u and v be any two different nodes in TQ_τ. By Corollary 1, the theorem holds when

$$\mathrm{dist}(TQ_\tau, u, v) = 1.$$

Therefore, by Lemma 4, we only need to consider the case for $2 \leq \mathrm{dist}(TQ_\tau, u, v) \leq \mathrm{diam}(TQ_\tau) = \lceil \frac{\tau+1}{2} \rceil$. For any $a, b \in \{0, 1\}$, without loss of generality, we deal with the following cases:

Case 1. $u, v \in V(TQ^{ab}_{\tau-2})$. For

$$\mathrm{dist}(TQ_\tau, u, v) + 2 \leq l \leq 2^\tau - 1,$$

we have the following subcases:

Case 1.1. $\mathrm{dist}(TQ_\tau, u, v) + 2 \leq l \leq 2^{\tau-2} - 1$. By Lemma 1, item 1, $\mathrm{dist}(TQ^{ab}_{\tau-2}, u, v) = \mathrm{dist}(TQ_\tau, u, v)$. By the induction hypothesis, there is a path of length l between u and v in $TQ^{ab}_{\tau-2}$ and, thus, in TQ_τ.

Case 1.2. $l = 2^{\tau-2}$. Let u' and v' be the neighbors, in $TQ^{(1-a)b}_{\tau-2}$, of u and v, respectively. By the induction hypothesis, there is a path P of length $2^{\tau-2} - 2$ between u' and v' in $TQ^{(1-a)b}_{\tau-2}$. Then,

$$u, P, v$$

is a path of length l between u and v in $TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2}$ and, thus, in TQ_τ.

Case 1.3. $2^{\tau-2} + 1 \leq l \leq 2^{\tau-1} - 1$. Let $l_1 = \lfloor \frac{l-1}{2} \rfloor$, $l_2 = (l-1) - l_1$. Then, $l_1 + l_2 = l - 1$ and, by Lemma 4,

$$\mathrm{dist}(TQ^{ab}_{\tau-2}, u, v) + 2 \leq \left\lceil \frac{\tau - 1}{2} \right\rceil + 2$$
$$\leq 2^{\tau-3} \leq l_1 \leq l_2 \leq 2^{\tau-2} - 1.$$

By the induction hypothesis, there is a path P_1 of length l_1 between u and v in $TQ^{ab}_{\tau-2}$ (See Fig. 4a). Select an edge (u', v') in the path P_1 and let u' be between u and v'. Further, respectively, select the neighbors u'' and v'', in $TQ^{(1-a)b}_{\tau-2}$, of the nodes u' and v'. By Definition 1, $(u'', v'') \in E(TQ^{(1-a)b}_{\tau-2})$. Since $4 \leq \lceil \frac{\tau-1}{2} \rceil + 2 \leq l_2 \leq 2^{\tau-2} - 1$, by Corollary 1, there is a path P_2 of length l_2 between u'' and v'' in $TQ^{(1-a)b}_{\tau-2}$. Then,

$$\mathrm{path}(P_1, u, u'), P_2, \mathrm{path}(P_1, v', v)$$

is a path of length $(l_1 - 1) + l_2 + 2 = l$ between u and v in $TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2}$ and, thus, in TQ_τ.

Case 1.4. $2^{\tau-1} \leq l \leq 2^\tau - 1$. Let $l_1 = \lfloor \frac{l-1}{2} \rfloor$ and $l_2 = (l-1) - l_1$. Then, $l_1 + l_2 = l - 1$ and

$$2^{\tau-2} - 1 \leq l_1 \leq l_2 \leq 2^{\tau-1} - 1.$$

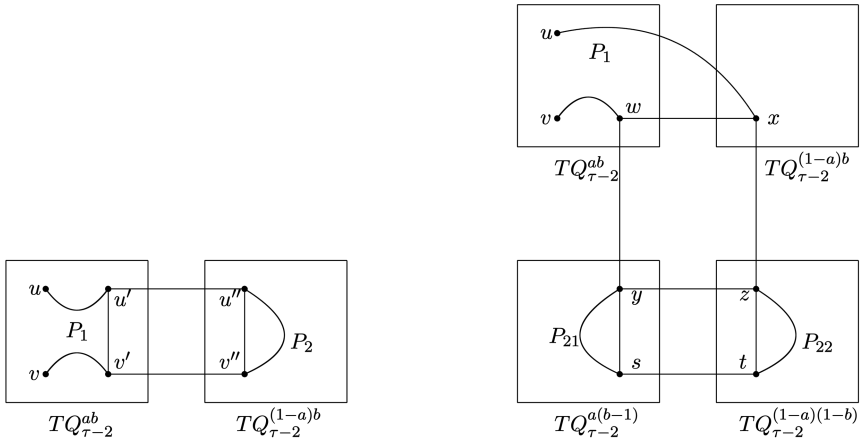
We first prove the following claim:

Claim. There is a path P_1 of length l_1 such that there is an edge (w, x) in P_1 with $w \in V(TQ^{ab}_{\tau-2})$ and $x \in V(TQ^{(1-a)b}_{\tau-2})$.

For $2^{\tau-2} \leq l_1 \leq 2^{\tau-1} - 1$, from Case 1.2 and Case 1.3, we can deduce that there is a path P_1 of length l_1 between u and v in $TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2}$. Since $l_1 \geq |V(TQ^{ab}_{\tau-2})|$, P_1 satisfies the conditions in the claim. For $l_1 = 2^{\tau-2} - 1$, let $l' = 2^{\tau-2} - 3$. Then,

$$\mathrm{dist}(TQ^{ab}_{\tau-2}, u, v) + 2 \leq \lceil \frac{\tau - 1}{2} \rceil$$
$$+ 2 \leq 2^{\tau-2} - 3 = l' \leq 2^{\tau-2} - 1.$$

By the induction hypothesis, there is a path P' of length l' between u and v in $TQ^{ab}_{\tau-2}$. Select an edge (u', v') in the



(a) (b)

Fig. 4. The path of length l between u and v in TQ_τ, where a straight line represents an edge and a curved line represents a path between two nodes.

path P_1 and let u' be between u and v'. Further, respectively, select the neighbors u'' and v'' in $TQ_{\tau-2}^{(1-a)b}$ of the nodes u' and v'. By Definition 1, $(u'', v'') \in E(TQ_{\tau-2}^{(1-a)b})$. Then, $\text{path}(P', u, u'), u'', v'', \text{path}(P_1, v', v)$ is a path of length $(l'-1)+3 = l_1$ between u and v in $TQ_{\tau-2}^{ab} \bigcup TQ_{\tau-2}^{(1-a)b}$. Let $w = u'$ and $x = u''$. Hence, the claim holds.

Now, we keep on with the following proof. By the above claim, let (w, x) be an edge in P_1 such that $w \in V(TQ_{\tau-2}^{ab})$ and $x \in V(TQ_{\tau-2}^{(1-a)b})$ (See Fig. 4b). Further, let y and z be the neighbors, in $TQ_{\tau-2}^{a(1-b)} \bigcup TQ_{\tau-2}^{(1-a)(1-b)}$, of w and x, respectively. By Definition 1, y and z lie in the different subcubes $TQ_{\tau-2}^{a(1-b)}$ and $TQ_{\tau-2}^{(1-a)(1-b)}$, respectively, with $(y, z) \in E(TQ_{\tau-2}^{a(1-b)} \bigcup TQ_{\tau-2}^{(1-a)(1-b)})$. Without loss of generality, we assume $y \in V(TQ_{\tau-2}^{a(1-b)})$ and $z \in V(TQ_{\tau-2}^{(1-a)(1-b)})$. Select an edge (y, s) in $TQ_{\tau-2}^{a(1-b)}$ and let t be the neighbor, in $TQ_{\tau-2}^{(1-a)(1-b)}$, of s. By Definition 1, $(z, t) \in E(TQ_{\tau-2}^{(1-a)(1-b)})$. Let $l_{21} = \lfloor \frac{l_2-1}{2} \rfloor$, $l_{22} = (l_2 - 1) - l_1$. Then, $l_{21} + l_{22} = l_2 - 1$ and $3 \le 2^{\tau-3} - 1 \le l_{21} \le l_{22} \le 2^{\tau-2} - 1$. By Corollary 1, there is a path P_{21} of length l_{21} between s and y in $E(TQ_{\tau-2}^{a(1-b)})$ and a path P_{22} of length l_{22} between z and t in $TQ_{\tau-2}^{(1-a)(1-b)}$. Then,

$$\text{path}(P_1, u, x), P_{22}, P_{21}, \text{path}(P_1, w, v),$$

is a path of length $(l_1 - 1) + l_{21} + l_{22} + 3 = l$ between u and v in TQ_τ.

Case 2. $u \in V(TQ_{\tau-2}^{ab})$ and $v \in V(TQ_{\tau-2}^{(1-a)b})$. For $\text{dist}(TQ_\tau, u, v) + 2 \le l \le 2^\tau - 1$, we have the following subcases:

Case 2.1. $\text{dist}(TQ_\tau, u, v) + 2 \le l \le 2^{\tau-2}$. By Lemma 1, item 2, without loss of generality, we can let P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$ be a shortest path between u and v in $TQ_{\tau-2}^{ab} \bigcup TQ_{\tau-2}^{(1-a)b}$ such that P satisfies the following three conditions for some integer $j \ge 0$ (See Fig. 5a):

1. $u^{(i)} \in V(TQ_{\tau-2}^{ab})$ for $i = 0, 1, \ldots, j$ with $0 \le j \le k-2$.
2. $u^{(i)} \in V(TQ_{\tau-2}^{(1-a)b})$ for $i = j+1, j+2, \ldots, k$.
3. $\text{len}(\text{path}(P, u^{(j+1)}, v)) \ge \text{len}(\text{path}(P, u, u^{(j)}))$.

Then, $k - j \ge 2$. Obviously, $\text{path}(P, u^{(j+1)}, v)$ is a shortest path between $u^{(j+1)}$ and v in both TQ_τ and $TQ_{\tau-2}^{(1-a)b}$. Since

$$\text{dist}\Big(TQ_{\tau-2}^{(1-a)b}, u^{(j+1)}, v\Big) + 2 = [\text{dist}(TQ_\tau, u, v) + 2] - \text{len}\Big(\text{path}\Big(P, u, u^{(j+1)}\Big)\Big) = [\text{dist}(TQ_\tau, u, v) + 2] - (j+1) \le l - (j+1) \le l - 1 \le 2^{\tau-2} - 1,$$

by the induction hypothesis, there is a path P' of length $l - (j+1)$ between $u^{(j+1)}$ and v in $TQ_{\tau-2}^{(1-a)b}$. Then,

$$\text{path}\Big(P, u, u^{(j)}\Big), P'$$

is a path of length $j + [l - (j+1)] + 1 = l$ between u and v in $TQ_{\tau-2}^{ab} \bigcup TQ_{\tau-2}^{(1-a)b}$ and, thus, in TQ_τ.

Case 2.2. $2^{\tau-2} + 1 \le l \le 2^{\tau-1} - 1$. We can always select an edge (x, y) in $TQ_{\tau-2}^{ab} \bigcup TQ_{\tau-2}^{(1-a)b}$ such that $x \in V(TQ_{\tau-2}^{ab}) - \{u\}$ and $y \in V(TQ_{\tau-2}^{(1-a)b}) - \{v\}$. Let $l_1 = \lfloor \frac{l-1}{2} \rfloor$, $l_2 = (l-1) - l_1$. Then, $l_1 + l_2 = l - 1$ and

$$\max\Big\{\text{dist}\Big(TQ_{\tau-2}^{ab}, u, x\Big), \text{dist}\Big(TQ_{\tau-2}^{(1-a)b}, v, y\Big)\Big\} + 2 \le \lceil \frac{\tau-1}{2} \rceil + 2 \le 2^{\tau-3} \le l_1 \le l_2 \le 2^{\tau-2} - 1.$$

By the induction hypothesis, there is a path P_1 of length l_1 between u and x in $TQ_{\tau-2}^{ab}$ and a path P_2 of length l_2 between y and v in $TQ_{\tau-2}^{(1-a)b}$. Then,

$$P_1, P_2$$

is a path of length $l_1 + l_2 + 1 = l$ between u and v in $TQ_{\tau-2}^{ab} \bigcup TQ_{\tau-2}^{(1-a)b}$ and, thus, in TQ_τ.

Case 2.3. $2^{\tau-1} \le l \le 2^\tau - 1$. Let $l_1 = \lfloor \frac{l-1}{2} \rfloor$, $l_2 = (l-1) - l_1$. Then, $l_1 + l_2 = l - 1$ and

$$2^{\tau-2} - 1 \le l_1 \le l_2 \le 2^{\tau-1} - 1.$$

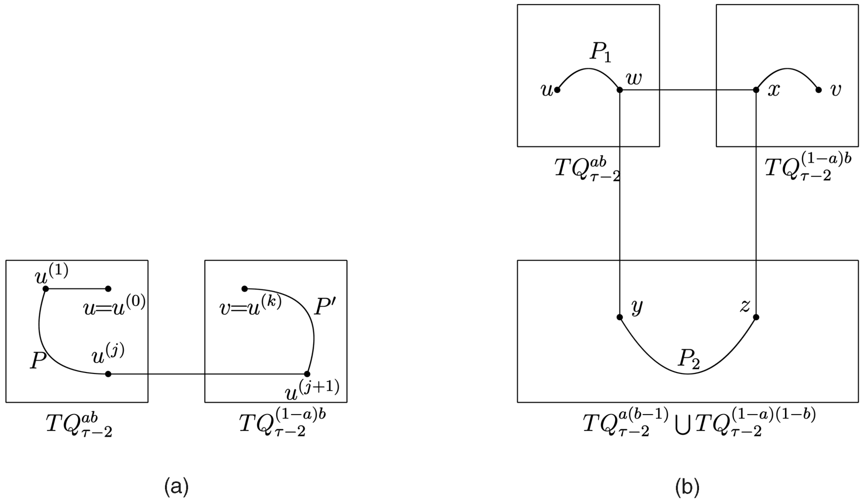


Fig. 5. The path of length l between u and v in TQ_τ, where a straight line represents an edge and a curve line represents a path between two nodes.

From Case 2.1 and Case 2.2, we can deduce that there is a path P_1 of length l_1 between u and v in $TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2}$ (See Fig. 5b). Clearly, there is an edge (w, x) in P_1 such that $w \in V(TQ^{ab}_{\tau-2})$ and $x \in V(TQ^{(1-a)b}_{\tau-2})$. Let y and z be the neighbors, in $TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}$, of w and x, respectively. By Definition 1, y and z are in $TQ^{a(1-b)}_{\tau-2}$ and $TQ^{(1-a)(1-b)}_{\tau-2}$, respectively. Still, from Case 2.1 and Case 2.2, we can deduce that there is a path P_2 of length l_2 between y and z in $TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}$. Then,

$$\mathrm{path}(P_1, u, w), P_2, \mathrm{path}(P_1, x, v)$$

is a path of length $(l_1 - 1) + l_2 + 2 = l$ between u and v in TQ_τ.

Case 3. $u \in V(TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2})$ and

$$v \in V\Big(TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}\Big).$$

For $\mathrm{dist}(TQ_\tau, u, v) + 2 \le l \le 2^\tau - 1$, we have the following subcases:

Case 3.1. $\mathrm{dist}(TQ_\tau, u, v) + 2 \le l \le 2^{\tau-1} - 1$. By Lemma 1, item 3, without loss of generality, we can let P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(k)} = v$ be a shortest path between u and v in TQ_τ such that P satisfies the following three conditions for some some integer $j \ge 0$ (Similar to Case 2.1):

1. $u^{(i)} \in V(TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2})$ for $i = 0, 1, \ldots, j$ with $0 \le j \le k-2$.
2. $u^{(i)} \in V(TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2})$ for $i = j+1, j+2, \ldots, k$.
3. $\mathrm{len}(\mathrm{path}(P, u^{(j+1)}, v)) \ge \mathrm{len}(\mathrm{path}(P, u, u^{(j)}))$.

Then, $k - j \ge 2$. Obviously, $\mathrm{path}(P, u^{(j+1)}, v)$ is a shortest path between $u^{(j+1)}$ and v in both TQ_τ and $TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}$. Considering that

$$u^{(j+1)} \in V\Big(TQ^{a(1-b)}_{\tau-2}\Big)$$

or $u^{(j+1)} \in V(TQ^{(1-a)(1-b)}_{\tau-2})$ and that $v \in V(TQ^{a(1-b)}_{\tau-2})$ or $v \in V(TQ^{(1-a)(1-b)}_{\tau-2})$, without loss of generality, we only need to consider the following two cases:

a. $u^{(j+1)}, v \in V(TQ^{a(1-b)}_{\tau-2})$ and

b. $u^{(j+1)} \in V(TQ^{a(1-b)}_{\tau-2})$ and $v \in V(TQ^{(1-a)(1-b)}_{\tau-2})$.

For Case 3.1.a, by Lemma 1, item 1,

$$\begin{aligned}&\mathrm{dist}\Big(TQ^{a(1-b)}_{\tau-2}, u^{(j+1)}, v\Big) + 2 = [\mathrm{dist}(TQ_\tau, u, v) + 2]\\ &- \mathrm{len}\Big(\mathrm{path}\Big(P, u, u^{(j+1)}\Big)\Big) = [\mathrm{dist}(TQ_\tau, u, v) + 2]\\ &- (j+1) \le l - (j+1) \le 2^{\tau-1} - 1.\end{aligned}$$

For Case 3.1.b, by Lemma 1, item 2,

$$\begin{aligned}&\mathrm{dist}\Big(TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}, u^{(j+1)}, v\Big) + 2\\ &= [\mathrm{dist}(TQ_\tau, u, v) + 2] - \mathrm{len}\Big(\mathrm{path}\Big(P, u, u^{(j+1)}\Big)\Big)\\ &\le l - (j+1) \le 2^{\tau-1} - 1.\end{aligned}$$

Thus, for Case 3.1.a by Case 1.1, Case 1.2, and Case 1.3 and for Case 3.1.b by Case 2.1 and Case 2.2, there is a path P' of length $l - (j+1)$ between $u^{(j+1)}$ and v in $TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}$. Then,

$$\mathrm{path}(P, u, u^{(j)}), P'$$

is a path of length $j + [l - (j+1)] + 1 = l$ between u and v in $TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2}$ and, thus, in TQ_τ.

Case 3.2. $2^{\tau-1} \le l \le 2^\tau - 1$. Clearly, we can always select an edge (x, y) such that $x \in V(TQ^{ab}_{\tau-2} \bigcup TQ^{(1-a)b}_{\tau-2}) - \{u\}$ and

$$y \in V\Big(TQ^{a(1-b)}_{\tau-2} \bigcup TQ^{(1-a)(1-b)}_{\tau-2}\Big) - \{v\}.$$

Let $l_1 = \lfloor \frac{l-1}{2} \rfloor$, $l_2 = (l-1) - l_1$. Then, $l_1 + l_2 = l - 1$ and $2^{\tau-2} - 1 \le l_1 \le l_2 \le 2^{\tau-1} - 1$. By Case 1 and Case 2, there is a path P_1 of length l_1 between u and x in

$TQ_{\tau-2}^{ab}\bigcup TQ_{\tau-2}^{(1-a)b}$ and a path P_2 of length l_2 between y and v in $TQ_{\tau-2}^{a(1-b)}\bigcup TQ_{\tau-2}^{(1-a)(1-b)}$. Then,

$$P_1, P_2$$

is a path of length $l_1+l_2+1=l$ between u and v in TQ_τ.

So, we have completed the proof when $n=\tau$. □

The proof of Theorem 2 is constructive. By this proof, we can find a path of length l between u and v for any $u,v\in V(TQ_n)$ with $u\neq v$ and any integer l with $\mathrm{dist}(TQ_n,u,v)+2\leq l\leq 2^n-1$ in TQ_n $(n\geq 3)$. For example, letting $n=5$, $u=00001$, $v=01100$, and $l=14$, the process to find a path of length l between u and v in TQ_n is as follows:

By Case 3.1, we can get a shortest path $u=u^{(0)}=00001$, $u^{(1)}=01001$, $u^{(2)}=01000$, $u^{(3)}=01100=v$ between u and v in TQ_5, which satisfies Conditions 1, 2, and 3 in Case 3.1, where $j=0$ and $k=3$. Since $8\leq l-(j+1)=13\leq 15$, let $l_1=\lfloor\frac{13-1}{2}\rfloor=6$ and $l_2=(13-1)-l_1=6$. By Case 1.3, we can select a path

$$P_1: u^{(1)}=01001, 01011, 01010, 01110, 01111, 01101, 01100=v$$

of length 6 between $u^{(1)}$ and v in TQ_3^{01}. Let $u'=01011$, $v'=01010$, $u''=11011$, and $v''=11010$. Note that (u',v') is an edge in the path P_1 and u'' and v'' are neighbors, in TQ_3^{11}, of u' and v', respectively. By Case 1.3, we can select a path

$$P_2: u''=11011, 11111, 11101, 11001, 11000, 11110, 11010=v''$$

of length 6 between u'' and v'' in TQ_3^{11}. Then,

$$u=00001, 01001, 01011, P_2, \mathrm{path}(P_1,u',v)$$

is a path of length $l=14$ between u and v in TQ_5.

4 Nonexistence and Existence of Embeddings of Paths with Length $\mathrm{dist}(TQ_n,u,v)+1$

Theorem 2 gives embeddings of paths of all possible lengths between $\mathrm{dist}(TQ_n,u,v)+2$ and 2^n-1 in TQ_n. In this section, we will discuss the cases for the nonexistence and existence of embeddings of paths with length $\mathrm{dist}(TQ_n,u,v)+1$. Theorems 3 and 4 discuss the nonexistence of and Theorems 5 and 6 discuss the existence of embeddings of paths with length $\mathrm{dist}(TQ_n,u,v)+1$, respectively. Before discussing these results, we first introduce the following two lemmas:

Lemma 5 [4]. *$h_d(u,v)\leq \mathrm{dist}(TQ_n,u,v)\leq h_d(u,v)+1$ for any $u,v\in V(TQ_n)$.*

Lemma 6. *For any $u,v\in V(TQ_n)$, if $h_d(u,v)=\lceil\frac{n+1}{2}\rceil$, then $\mathrm{dist}(TQ_n,u,v)=\lceil\frac{n+1}{2}\rceil$.*

Proof. If $h_d(u,v)=\lceil\frac{n+1}{2}\rceil$, by Lemma 5, $\mathrm{dist}(TQ_n,u,v)\geq h_d(u,v)=\lceil\frac{n+1}{2}\rceil$. By Lemma 4, $\mathrm{dist}(TQ_n,u,v)\leq\lceil\frac{n+1}{2}\rceil$. Hence, $\mathrm{dist}(TQ_n,u,v)=\lceil\frac{n+1}{2}\rceil$. □

Theorem 3. *For any $n\geq 1$, a cycle of length 3 cannot be embedded with dilation 1 in TQ_n.*

Proof. We can easily verify that the result holds for $n\in\{1,3\}$. Then, we will prove the following two claims:

1. For $n\geq 3$, if there is no cycle of length 3 in TQ_n, then there is no a cycle of length 3 in $TQ_n^{ab}\bigcup TQ_n^{(1-a)b}$ for any $a,b\in\{0,1\}$.
2. For $n\geq 5$, if there is no cycle of length 3 in $TQ_{n-2}^{ab}\bigcup TQ_{n-2}^{(1-a)b}$ for any $a,b\in\{0,1\}$, then there is no a cycle of length 3 in TQ_n.

The two claims can be proved similar to the proof of Lemma 5 in [8]. Hence, we omit the further proof. □

Theorem 4. *For any $n\geq 3$ and any l with $2\leq l\leq\lceil\frac{n+1}{2}\rceil-1$, there exist two nodes $u,v\in V(TQ_n)$ such that $\mathrm{dist}(TQ_n,u,v)=l$ and a path of length $l+1$ cannot be embedded between u and v with dilation 1 in TQ_n.*

Proof. For $2\leq l\leq\lceil\frac{n+1}{2}\rceil-1$ and $n\geq 3$, let $u=(11)^l0^{n-2l}$ and $v=0^n$.

First, we prove $\mathrm{dist}(TQ_n,u,v)=l$. Let

$$u^{(j)}=(00)^j(11)^{l-j}0^{n-2l}, j=0,1,\ldots,l.$$

By the AP algorithm, we can verify that $u^{(0)}=u, u^{(1)},\ldots,u^{(l)}=v$ is a shortest path between u and v in TQ_n. Obviously, the length of this path is l. Hence, $\mathrm{dist}(TQ_n,u,v)=l$.

In what follows, we will prove that there does not exist a path of length $l+1$ between u and v in TQ_n by contradiction.

Suppose that there is a path, say

$$P: u=u^{(0)}, u^{(1)}, \ldots, u^{(l+1)}=v,$$

of length $l+1$ between u and v in TQ_n. Let $u^{(k)}=u_{n-1}^{(k)}u_{n-2}^{(k)}\ldots u_1^{(k)}u_0^{(k)}$ for $k=0,1,\ldots,l+1$.

Since $(u^{(0)},u^{(1)})\in E(TQ_n)$, by Definition 1, there is a double bit such that $u^{(1)}$ differs from $u^{(0)}$ in this double bit. Assume that $u^{(1)}$ differs from $u^{(0)}$ in some j_1th double bit with $0\leq j_1\leq\frac{n-2l-1}{2}$. Noticing that $f(u^{(0)},i')=0$ for any $i'\in\{0,2,4,\ldots,n-3\}$, we have $u_0^{(1)}=\overline{u_0^{(0)}}=1$ if $j_1=0$ and $u_{2j_1}^{(1)}u_{2j_1-1}^{(1)}=11$ if $1\leq j_1\leq\frac{n-2l-1}{2}$. Then,

$$h_d(u^{(1)},v)=h_d(u^{(0)},v)+1=l+1.$$

By Lemma 5, $\mathrm{dist}(TQ_n,u^{(1)},v)\geq h_d(u^{(1)},v)\geq l+1$ and

$$\begin{aligned}\mathrm{len}(P)&=\mathrm{len}(\mathrm{path}(P,u,u^{(1)}))+\mathrm{len}(\mathrm{path}(P,u^{(1)},v))\\&\geq 1+\mathrm{dist}(TQ_n,u^{(1)},v)\geq l+2.\end{aligned}$$

This contradicts that the length of the path P is $l+1$. Therefore, $\frac{n-2l+1}{2}\leq j_1\leq\frac{n-1}{2}$. We have the following two cases:

1. $u_{2j_1}^{(1)}u_{2j_1-1}^{(1)}=\overline{u_{2j_1}^{(0)}}\,\overline{u_{2j_1-1}^{(0)}}=00=u_{2j_1}^{(l+1)}u_{2j_1-1}^{(l+1)}$. Then, $u_{i'}^{(1)}=u_{i'}^{(0)}$ for all
$$i'\in\{0,1,\ldots,n-1\}-\{2j_1-1,2j_1\}.$$
Thus, $h_d(u^{(1)},v)=h_d(u^{(0)},v)-1=l-1$.
2. $u_{2j_1}^{(1)}u_{2j_1-1}^{(1)}=\overline{u_{2j_1}^{(0)}}u_{2j_1-1}^{(0)}=01=u_{2j_1}^{(l+1)}\overline{u_{2j_1-1}^{(l+1)}}$. Then, $u_{i'}^{(1)}=u_{i'}^{(0)}$ for all
$$i'\in\{0,1,\ldots,n-1\}-\{2j_1-1,2j_1\}.$$
Thus, $h_d(u^{(1)},v)=h_d(u^{(0)},v)=l$.

For $1 \le m \le l-2$, suppose that there are m integers $j_1, j_2, \ldots, j_m$ such that one of the following two cases holds:

Case 1. $u^{(m)}$ satisfies the following three conditions:

1. $u^{(m)}_{i'} = 0$ for all $i' \in \{0, 1, \ldots, n-2l-1\}$.
2. $\frac{n-2l+1}{2} \le j_k \le \frac{n-1}{2}$ and $u^{(m)}_{2j_k} u^{(m)}_{2j_k-1} = 00 = u^{(l+1)}_{2j} u^{(l+1)}_{2j-1}$ for all $k \in \{1, 2, \ldots, m\}$.
3. $u^{(m)}_{2j_{k'}} u^{(m)}_{2j_{k'}-1} = 11$ for all

$$k' \in \{\frac{n-2l+1}{2}, \frac{n-2l+3}{2}, \ldots, \frac{n-1}{2}\} - \{j_1, j_2, \ldots, j_m\}.$$

Case 2. $u^{(m)}$ satisfies the following four conditions:

1. $u^{(m)}_{i'} = 0$ for all $i' \in \{0, 1, \ldots, n-2l-1\}$.
2. There is an integer q such that $\frac{n-2l+1}{2} \le j_q \le \frac{n-1}{2}$ and $u^{(m)}_{2j_q} u^{(m)}_{2j_q-1} = 01 = u^{(l+1)}_{2j_q} \overline{u^{(l+1)}_{2j_q-1}}$, where q is an integer with $1 \le q \le m$.
3. $\frac{n-2l+1}{2} \le j_k \le \frac{n-1}{2}$ and $u^{(m)}_{2j_k} u^{(m)}_{2j_k-1} = 00$ for all $k \in \{1, 2, \ldots, m\} - \{q\}$.
4. $u^{(m)}_{2j_{k'}} u^{(m)}_{2j_{k'}-1} = 11$ for all

$$k' \in \left\{\frac{n-2l+1}{2}, \frac{n-2l+3}{2}, \ldots, \frac{n-1}{2}\right\} - \{j_1, j_2, \ldots, j_m\}.$$

Then, we separately discuss Case 1 and Case 2 in the following:

For Case 1, we have

$$h_d(u^{(m)}, v) = h_d(u^{(0)}, v) - m = l - m.$$

For $u^{(m+1)}$, since $(u^{(m)}, u^{(m+1)}) \in E(TQ_n)$, by Definition 1, there is a double bit such that $u^{(m+1)}$ differs from $u^{(m)}$ in this double bit. Suppose that $u^{(m+1)}$ differs from $u^{(m)}$ in some j_{m+1}th double bit with $0 \le j_{m+1} \le \frac{n-1}{2}$. Similar to the above discussion about $u^{(0)}$ and $u^{(1)}$, it is not possible that $0 \le j_{m+1} \le \frac{n-2l-1}{2}$. Hence, $\frac{n-2l+1}{2} \le j_{m+1} \le \frac{n-1}{2}$. Further, we have $j_{m+1} \notin \{j_1, j_2, \ldots, j_m\}$. Otherwise, $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} \ne u^{(l+1)}_{2j_{m+1}} u^{(l+1)}_{2j_{m+1}-1}$ and $u^{(m+1)}_{i'} = u^{(m)}_{i'}$ for all

$$i' \in \{0, 1, \ldots, n-1\} - \{2j_{m+1}-1, 2j_{m+1}\}.$$

Thus, $h_d(u^{(m+1)}, v) = h_d(u^{(m)}, v) + 1 = l - m + 1$. By Lemma 5,

$$\begin{aligned}\mathrm{len}(P) &= \mathrm{len}\Big(\mathrm{path}\Big(P, u, u^{(m+1)}\Big)\Big) + \mathrm{len}\Big(\mathrm{path}\Big(P, u^{(m+1)}, v\Big)\Big) \\ &\ge (m+1) + \mathrm{dist}(TQ_n, u^{(m+1)}, v) \ge (m+1) \\ &\quad + h_d(u^{(m+1)}, v) \ge (m+1) + (l-m+1) = l+2,\end{aligned}$$

contradicting that the length of the path P is $l+1$. Hence, $j_{m+1} \notin \{j_1, j_2, \ldots, j_m\}$. As a result, we have $\frac{n-2l+1}{2} \le j_{m+1} \le \frac{n-1}{2}$ such that $u^{(m)}_{2j_{m+1}} u^{(m)}_{2j_{m+1}-1} = 11$. Thus, we have the following two cases for Case 1:

a. $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} = \overline{u^{(m)}_{2j_{m+1}}}\ \overline{u^{(m)}_{2j_{m+1}-1}} = 00 = u^{(l+1)}_{2j_{m+1}} u^{(l+1)}_{2j_{m+1}-1}$. By Definition 1, $u^{(m+1)}_{i'} = u^{(m)}_{i'}$ for all

$$i' \in \{0, 1, \ldots, n-1\} - \{2j_{m+1}-1, 2j_{m+1}\}.$$

Thus, $h_d(u^{(m+1)}, v) = h_d(u^{(m)}, v) - 1 = l - (m+1)$.

b. $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} = \overline{u^{(m)}_{2j_{m+1}}} u^{(m)}_{2j_{m+1}-1} = 01 = u^{(l+1)}_{2j_{m+1}} \overline{u^{(l+1)}_{2j_{m+1}-1}}$. By Definition 1, $u^{(m+1)}_{i'} = u^{(m)}_{i'} q$ for all

$$i' \in \{0, 1, \ldots, n-1\} - \{2j_{m+1}-1, 2j_{m+1}\}.$$

Thus, $h_d(u^{(m+1)}, v) = h_d(u^{(m)}, v) = l - m$.

For Case 2, we have

$$h_d(u^{(m)}, v) = h_d(u^{(0)}, v) - (m-1) = l - m + 1.$$

For $u^{(m+1)}$, since $(u^{(m)}, u^{(m+1)}) \in E(TQ_n)$, by Definition 1, there is a double bit such that $u^{(m+1)}$ differs from $u^{(m)}$ in this double bit. Suppose that $u^{(m+1)}$ differs from $u^{(m)}$ in some j_{m+1}th double bit with $0 \le j_{m+1} \le \frac{n-1}{2}$. Similar to the above discussion about $u^{(0)}$ and $u^{(1)}$, it is not possible that $0 \le j_{m+1} \le \frac{n-2l-1}{2}$. Hence, $\frac{n-2l+1}{2} \le j_{m+1} \le \frac{n-1}{2}$.

In what follows, we will prove that

$$j_{m+1} \notin \{j_1, j_2, \ldots, j_m\}.$$

First, we have $j_{m+1} \ne j_q$. Otherwise, by Definition 1, $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} \in \{11, 10\}$ and, hence,

$$u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} \ne u^{(l+1)}_{2j_{m+1}} u^{(l+1)}_{2j_{m+1}-1}.$$

Similar to the discussion in Case 1, we can deduce $h_d(u^{(m+1)}, v) = h_d(u^{(m)}, v) = l - m + 1$ and $\mathrm{len}(P) \ge l+2$, contradicting that the length of the path P is $l+1$. Therefore, $j_{m+1} \ne j_q$. Further, also similar to the discussion in Case 1, we can deduce $j_{m+1} \notin \{j_1, j_2, \ldots, j_m\} - \{j_q\}$. To sum up, $j_{m+1} \notin \{j_1, j_2, \ldots, j_m\}$.

As a result, we have $u^{(m)}_{2j_{m+1}} u^{(m)}_{2j_{m+1}-1} = 11$ and

$$\frac{n-2l+1}{2} \le j_{m+1} \le \frac{n-1}{2}.$$

By Definition 1, $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} = \overline{u^{(m)}_{2j_{m+1}}}\ \overline{u^{(m)}_{2j_{m+1}-1}} = 00 = u^{(l+1)}_{2j_{m+1}} u^{(l+1)}_{2j_{m+1}-1}$ and $u^{(m+1)}_{i'} = u^{(m)}_{i'}$ for all

$$i' \in \{0, 1, \ldots, n-1\} - \{2j_{m+1}-1, 2j_{m+1}\}.$$

Thus, $h_d(u^{(m+1)}, v) = h_d(u^{(m)}, v) - 1 = l - m$.

According to the discussions for Cases 1 and 2, we have the following two cases:

Case A. For any m with $0 \le m \le l-2$, there exists an integer j_{m+1} with $\frac{n-2l+1}{2} \le j_{m+1} \le \frac{n-1}{2}$ such that $u^{(m+1)}$ differs from $u^{(m)}$ in the j_{m+1}th double bit, $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} = \overline{u^{(m)}_{2j_{m+1}}}\ \overline{u^{(m)}_{2j_{m+1}-1}} = 00$, and $u^{(m+1)}_{i'} = u^{(m)}_{i'}$ for all $i' \in \{0, 1, \ldots, n-1\} - \{2j_{m+1}-1, 2j_{m+1}\}$. Then, we have $h_d(u^{(l-1)}, v) = 1$ and there is an integer i'' such that $u^{(l-1)} = (00)^{i''}(11)0^{n-2i''-2}$. In summary, $(u^{(l-1)}, v) \in E(TQ_n)$. Considering that

$$P : u = u^{(0)}, u^{(1)}, \ldots, u^{(l+1)} = v$$

is a path, we have $(u^{(l-1)}, u^{(l)}), (u^{(l)}, u^{(l+1)}) \in E(TQ_n)$ and, thus, $u^{(l-1)}, u^{(l)}, u^{(l+1)} = v, u^{(l-1)}$ is a cycle of length 3, contradicting Theorem 3.

Case B. There exists an integer m with $0 \le m \le l-1$ such that the following two conditions hold:

1. $u^{(m+1)}$ differs from $u^{(m)}$ in the j_{m+1}th double bit for some integer j_{m+1} with $\frac{n-2l+1}{2} \le j_{m+1} \le \frac{n-1}{2}$, $u^{(m+1)}_{2j_{m+1}} u^{(m+1)}_{2j_{m+1}-1} = \overline{u^{(m)}_{2j_{m+1}} u^{(m)}_{2j_{m+1}-1}} = 01$, and $u^{(m+1)}_{i'} = u^{(m)}_{i'}$ for $i' \in \{0, 1, \ldots, n-1\} - \{2j_{m+1}\}$.
2. For any $m' \in \{0, 1, \ldots, l-1\} - \{m\}$, there exists an integer $j_{m'+1}$ with $\frac{n-2l+1}{2} \le j_{m'+1} \le \frac{n-1}{2}$ such that $u^{(m'+1)}$ differs from $u^{(m')}$ in the $j_{m'+1}$th double bit, $u^{(m'+1)}_{2j_{m'+1}} u^{(m'+1)}_{2j_{m'+1}-1} = \overline{u^{(m')}_{2j_{m'+1}} u^{(m')}_{2j_{m'+1}-1}} = 00$, and $u^{(m'+1)}_{i'} = u^{(m')}_{i'}$ for all
$$i' \in \{0, 1, \ldots, n-1\} - \{2j_{m'+1} - 1, 2j_{m'+1}\}.$$

Then, we have $h_d(u^{(l)}, v) = 1$ and there is an integer i'' such that $u^{(l)} = (00)^{i''}(01)0^{n-2i''-1}$. Since
$$f(u^{(l)}, n - 2i'' - 2) = 0,$$
by Definition 1, $(u^{(l)}, v) \notin E(TQ_n)$. However, since P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(l)}, u^{(l+1)} = v$ is a path, we should have $(u^{(l)}, v) \in E(TQ_n)$, a contradiction.

According to the above discussion, there does not exist a path of length $l+1$ between u and v in TQ_n. □

Theorem 5. *For any $n \ge 3$ and any l with $2 \le l \le \lceil \frac{n+1}{2} \rceil - 1$, there exist two nodes u, $v \in V(TQ_n)$ such that*
$$\mathrm{dist}(TQ_n, u, v) = l$$
and the path of length $l+1$ can be embedded between u and v with dilation 1 in TQ_n.

Proof. Let $u = (11)^{l-1}0^{n-2l+1}1$ and $v = 0^n$. Then, $h_d(u, v) = l$.

First, we prove $\mathrm{dist}(TQ_n, u, v) = l$. Let
$$u^{(1)} = (11)^{l-1}0^{n-2l+2},$$
$u^{(i)} = 0^{2i-2}(11)^{l-i}0^{n-2l+2}$, $i = 2, 3, \ldots, l$. By Definition 1, we can easily verify that P: $u = u^{(0)}, u^{(1)}, \ldots, u^{(l)} = v$ is a path of length l between u and v in TQ_n. Hence, $\mathrm{dist}(TQ_n, u, v) \le \mathrm{len}(P) = l$. By Lemma 5,
$$\mathrm{dist}(TQ_n, u, v) \ge h_d(u, v) = l.$$
Consequently, $\mathrm{dist}(TQ_n, u, v) = l$.

Next, we prove that there is a path of length $l+1$ between u and v in TQ_n. Let $v^{(1)} = (11)^{l-2}(01)0^{n-2l+1}1$, $v^{(2)} = (11)^{l-2}0^{n-2l+3}1$, $v^{(3)} = (11)^{l-2}0^{n-2l+4}$,
$$v^{(i)} = (11)^{l-i+1}0^{n-2l+2i-2},$$
$i = 4, 5, \ldots, l+1$. By Definition 1, we can easily verify that P: $u = v^{(0)}, v^{(1)}, \ldots, v^{(l+1)} = v$ is a path of length $l+1$ between u and v in TQ_n. □

Theorem 6. *For any $n \ge 3$ and any $u, v \in V(TQ_n)$, if $\mathrm{dist}(TQ_n, u, v) = \lceil \frac{n+1}{2} \rceil$, then there is a path of length $\lceil \frac{n+1}{2} \rceil + 1$ between u and v in TQ_n.*

Proof. We still use induction on n.

By Lemma 2, TQ_3 is isomorphic to CQ_3. Further, by Lemma 3, when $n = 3$, the theorem holds.

Supposing that the theorem holds for $n = \tau - 2 (\tau \ge 5)$, we consider the case for $n = \tau$.

Let u and v be any two nodes with $\mathrm{dist}(TQ_\tau, u, v) = \lceil \frac{\tau+1}{2} \rceil$ in TQ_τ.

Clearly, for any $a, b \in \{0, 1\}$, both u and v are not in $TQ^{ab}_{\tau-2}$. Without loss of generality, we separately deal with the following cases:

Case 1. There exist $a, b, c, d, \in \{0, 1\}$ and
$$w \in V(TQ^{cd}_{\tau-2}),$$
such that $u \in V(TQ^{ab}_{\tau-2})$, $v \in V(TQ^{cd}_{\tau-2})$, $(u, w) \in E(TQ_\tau)$, and $ab \ne cd$.

By the AP algorithm, w is in a shortest path between u and v in TQ_τ and
$$\mathrm{dist}(TQ_\tau, w, v) = \mathrm{dist}(TQ_\tau, u, v) - \mathrm{dist}(TQ_\tau, u, w) = \left\lceil \frac{\tau+1}{2} \right\rceil - 1 = \left\lceil \frac{\tau-1}{2} \right\rceil.$$
Considering that $w, v \in V(TQ^{cd}_{\tau-2})$, by Lemma 1, item 1, $\mathrm{dist}(TQ^{cd}_{\tau-2}, w, v) = \mathrm{dist}(TQ_\tau, w, v) = \lceil \frac{\tau-1}{2} \rceil$. By the induction hypothesis, there is a path P of length $\lceil \frac{\tau-1}{2} \rceil + 1$ between w and v in $TQ^{cd}_{\tau-2}$. Then,
$$u, P$$
is a path of length $\lceil \frac{\tau-1}{2} \rceil + 2 = \lceil \frac{\tau+1}{2} \rceil + 1$ between u and v in TQ_τ.

Case 2. For any $a, b, c, d \in \{0, 1\}$ with $ab \ne cd$, all of the following three conditions hold:

1. $u \in V(TQ^{ab}_{\tau-2})$, $v \in V(TQ^{cd}_{\tau-2})$.
2. For any $s \in V(TQ^{cd}_{\tau-2})$, $(u, s) \notin E(TQ_\tau)$.
3. For any $y \in V(TQ^{ab}_{\tau-2})$, $(v, y) \notin E(TQ_\tau)$.

For simplicity, we only consider the case for $u \in V(TQ^{00}_{\tau-2})$, $v \in V(TQ^{01}_{\tau-2})$. For other cases, similarly discuss. By Definition 1, there exists a node $w \in V(TQ^{11}_{\tau-2})$ such that $(u, w) \in E(TQ_\tau)$. By Lemma 5, we deal with the following subcases.

Case 2.1. $\mathrm{dist}(TQ_\tau, u, v) = h_d(u, v)$. By Definition 1, we can let x be the neighbor, in $TQ^{11}_{\tau-2}$, of v (see Fig. 6a). Then, $\mathrm{bit}(x, i) = \mathrm{bit}(v, i)$ for all $i \in \{0, 1, \ldots, \tau-3\}$. Similarly, since $w \in V(TQ^{11}_{\tau-2})$ and $(u, w) \in E(TQ_\tau)$, by Definition 1, $\mathrm{bit}(w, i) = \mathrm{bit}(u, i)$ for all $i \in \{0, 1, \ldots, \tau-3\}$. Obviously, $\mathrm{bit}(w, i) = \mathrm{bit}(x, i)$ for $i \in \{\tau-1, \tau-2\}$. Therefore, $h_d(w, x) = h_d(u, v) - 1 = \lceil \frac{\tau+1}{2} \rceil - 1 = \lceil \frac{(\tau-2)+1}{2} \rceil$. By Lemma 6,
$$\mathrm{dist}(TQ_\tau, w, x) = h_d(w, x) = \left\lceil \frac{(\tau-2)+1}{2} \right\rceil = \left\lceil \frac{\tau-1}{2} \right\rceil.$$
By Lemma 1, item 1, $\mathrm{dist}(TQ^{11}_{\tau-2}, w, x) = \mathrm{dist}(TQ_\tau, w, x) = \lceil \frac{\tau-1}{2} \rceil$. Hence, we can let P be a path of length $\lceil \frac{\tau-1}{2} \rceil$ between w and x in $TQ^{11}_{\tau-2}$. Then,
$$u, P, v$$
is a path of length $\lceil \frac{\tau-1}{2} \rceil + 2 = \lceil \frac{\tau+1}{2} \rceil + 1$ between u and v in TQ_τ.

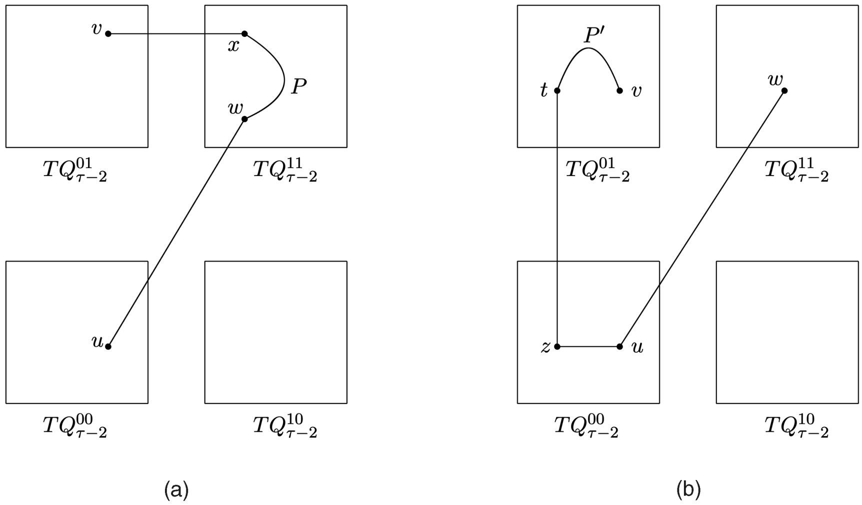


Fig. 6. The path of length $\lceil\frac{\tau+1}{2}\rceil + 1$ between u and v in TQ_τ, where a straight line represents an edge and a curve line represents a path between two nodes.

Case 2.2. $\mathrm{dist}(TQ_\tau, u, v) = h_d(u, v) + 1$. Then, there is a unique same double bit between u and v. We assume the jth double bit is this same double bit between u and v. Clearly, $0 \le j \le \lceil\frac{n-1}{2}\rceil$. Let $u = u_{n-1}u_{n-2}\dots u_0$. Further, let $z = u_{n-1}u_{n-2}\dots u_{2j+1}\overline{u_{2j}}u_{2j-1}u_{2j-2}\dots u_0$ if $1 \le j \le \lceil\frac{n-1}{2}\rceil$ and $z = u_{n-1}u_{n-2}\dots u_2u_1\overline{u_0}$ if $j = 0$. By Definition 1, $(u, z) \in V(TQ^{00}_{\tau-2})$ and

$$h_d(z, v) = h_d(u, v) + 1 = \mathrm{dist}(TQ_\tau, u, v) = \lceil\tfrac{\tau+1}{2}\rceil$$

(see Fig. 6b). Since Condition 2 holds, there exists a $w \in V(TQ^{11}_{\tau-2})$ such that $(u, w) \in E(TQ_\tau)$. By Definition 1, $f(z, \tau-3) = 1 - f(u, \tau-3) = 1$. Thus, z has a neighbor in $TQ^{01}_{\tau-2}$. Let the neighbor, in $TQ^{01}_{\tau-2}$, of z be t. Then,

$$\begin{aligned} h_d(t, v) &= h_d(z, v) - 1 = h_d(u, v) = \mathrm{dist}(TQ_\tau, u, v) - 1 \\ &= \left\lceil\frac{\tau-1}{2}\right\rceil = \left\lceil\frac{(\tau-2)+1}{2}\right\rceil. \end{aligned}$$

By Lemma 6, $\mathrm{dist}(TQ^{00}_{\tau-2}, t, v) = h_d(t, v) = \lceil\frac{\tau-1}{2}\rceil$. Let P' be a path of length $\lceil\frac{\tau-1}{2}\rceil$ between t and v in $TQ^{00}_{\tau-2}$. Then,

$$u, z, P'$$

is a path of length $\lceil\frac{\tau-1}{2}\rceil + 2 = \lceil\frac{\tau+1}{2}\rceil + 1$ between u and v in TQ_τ. □

5 Conclusions

Twisted cubes are variants of hypercubes. In this paper, we have studied the optimal embeddings of paths of all possible lengths between arbitrary two distinct nodes with dilation 1 in twisted cubes. We have proved the following desirable results of TQ_n: 1) For any two distinct nodes u and v and any integer l with $\mathrm{dist}(TQ_n, u, v) + 2 \le l \le 2^n - 1$, a path of length l can be embedded between u and v with dilation 1 ($n \ge 3$). 2) There exist two nodes u and v such that no path of length $\mathrm{dist}(TQ_n, u, v) + 1$ can be embedded between u and v with dilation 1 ($n \ge 3$). The special cases for the nonexistence and existence of embeddings of paths between nodes u and v and with length $\mathrm{dist}(TQ_n, u, v) + 1$ have also been discussed.

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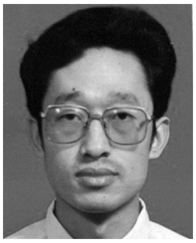

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