Strong Converse Theorems for Classes of Multimessage Multicast Networks: A Rényi Divergence Approach

Silas L. Fong and Vincent Y. F. Tan

Department of Electrical and Computer Engineering, National University of Singapore, Emails: {silas_fong,vtan}@nus.edu.sg

***Abstract*—This work establishes the strong converse theorems for some classes of discrete memoryless multimessage multicast networks whose corresponding cut-set bounds are tight.**

I. Introduction

This paper considers multimessage multicast networks (MMNs) in which the destination nodes want to decode the same set of messages transmitted by the source nodes. A well-known outer bound on the capacity region of the discrete memoryless MMN (DM-MMN) is the *cut-set bound*, developed by El Gamal in 1981. It is known that the cut-set bound is not tight in general, but it is tight for several classes of DM-MMNs, including the deterministic relay network with no interference [1], the finite-field linear deterministic network [2] and the wireless erasure network [3].

Although weak converse theorems for the above three networks have been established, strong converse theorems are lacking. Strong converse implies that all codes whose error probabilities are no larger than $\epsilon \in [0, 1)$ as the block length grows, i.e., ϵ-reliable codes, must have rate tuples belonging to the region prescribed by the fundamental limit. This is clearly a stronger statement than the weak converse which considers codes with vanishing error probabilities.

The main contribution of this work is proving the strong converse for some classes of DM-MMNs, including the three networks mentioned above. The technique that we employ is based on properties of the Rényi divergence. It has been employed previously to establish strong converses for DMCs with output feedback [4], classical-quantum channels and most recently, entanglement-breaking quantum channels.

II. Summary of Results

We consider a DM-MMN that consists of N nodes. Let $\mathcal{I} \triangleq \{1, 2, \ldots, N\}$ be the index set of the nodes, and let $\mathcal{S} \subseteq \mathcal{I}$ and $\mathcal{D} \subseteq \mathcal{I}$ be the sets of sources and destinations respectively. We call $(\mathcal{S}, \mathcal{D})$ the *multicast demand* on the network. The sources in $\mathcal{S}$ transmit information to the destinations in $\mathcal{D}$ in n time slots as follows. Each node $i \in \mathcal{S}$ transmits message W_i, uniformly distributed on $\{1, 2, \ldots, \lceil 2^{nR_i} \rceil\}$, and each node $j \in \mathcal{D}$ wants to decode $\{W_i : i \in \mathcal{S}\}$, where R_i denotes the rate of message W_i. Wlog, we assume $R_i = 0$ for all $i \in \mathcal{S}^c$. For each time slot $k \in \{1, 2, \ldots, n\}$ and each $i \in \mathcal{I}$, node i transmits $X_{i,k} \in \mathcal{X}_i$, a function of (W_i, Y_i^{k-1}), and receives, from the output of a channel, $Y_{i,k} \in \mathcal{Y}_i$ where $\mathcal{X}_i$ and $\mathcal{Y}_i$ are the alphabets of the symbols transmitted by i and received by i respectively. After n time slots, node j declares $\hat{W}_{i,j}$ to be the transmitted W_i based on (W_j, Y_j^n) for each $(i, j) \in \mathcal{S} \times \mathcal{D}$.

To simplify notation, we use the following conventions for each $T \subseteq \mathcal{I}$: For any N-dimensional random tuple $(X_1, X_2, \ldots, X_N)$, we let $X_T \triangleq (X_i : i \in T)$ be a subtuple of the random tuple. Then, we let $q_{Y_\mathcal{I}|X_\mathcal{I}}$ be a transition matrix that characterizes the channel of the DM-MMN such that the following holds for each time slot k: $\mathbf{Pr}\{W_\mathcal{I} = w_\mathcal{I}, X_\mathcal{I}^k = x_\mathcal{I}^k, Y_\mathcal{I}^k = y_\mathcal{I}^k\} =$ $\mathbf{Pr}\{W_\mathcal{I} = w_\mathcal{I}, X_\mathcal{I}^k = x_\mathcal{I}^k, Y_\mathcal{I}^{k-1} = y_\mathcal{I}^{k-1}\} q_{Y_\mathcal{I}|X_\mathcal{I}}(y_{\mathcal{I},k}|x_{\mathcal{I},k})$.

We define an $(n, R_\mathcal{I})$-code for the DM-MMN as a length-n code whose message rates are characterized by $R_\mathcal{I} \triangleq (R_1, R_2, \ldots, R_N)$. The error probability of the $(n, R_\mathcal{I})$-code is defined to be $\mathbf{Pr}\left\{\bigcup_{j\in\mathcal{D}} \bigcup_{i\in\mathcal{S}} \left\{\hat{W}_{i,j} \neq W_i\right\}\right\}$. The $(n, R_\mathcal{I})$-code is said to be ϵ_n*-reliable* if its error probability is no larger than ϵ_n. For each $\epsilon \in [0, 1)$, a rate tuple $R_\mathcal{I}$ is said to be ϵ*-achievable* if there exists a sequence of ϵ_n-reliable $(n, R_\mathcal{I})$-codes such that $\lim_{n\to\infty} \epsilon_n \leq \epsilon$. We define the ϵ-capacity region, denoted by $\mathcal{C}_\epsilon$, to be the set of ϵ-achievable rate tuples. The following is our main theorem.

Theorem. *For each $\epsilon \in [0, 1)$, $\mathcal{C}_\epsilon$ is contained in*

$$\bigcap_{\substack{T\subseteq\mathcal{I}:\\ T^c\cap\mathcal{D}\neq\emptyset}} \bigcup_{p_{X_\mathcal{I}}} \left\{ R_\mathcal{I} \,\middle|\, \begin{array}{l} \sum_{i\in T} R_i \leq I_{p_{X_\mathcal{I}} q_{Y_{T^c}|X_\mathcal{I}}}(X_T; Y_{T^c}|X_{T^c}). \\ R_i = 0 \text{ for all } i \in \mathcal{S}^c \end{array} \right\},$$

This region is achievable for the three classes of networks in [1]–[3] so the strong converse holds for these networks.

Notice that the set above is similar to the cut-set bound except that the intersection and union are interchanged. The proof of the theorem uses the Rényi divergence approach (e.g., [4]), and it can be found in the extended version of this paper [5].

References

[1] N. Ratnakar and G. Kramer, "The multicast capacity of deterministic relay networks with no interference," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2425–2432, 2006.

[2] A. S. Avestimehr, S. N. Diggavi, and D. N. Tse, "Wireless network information flow: A deterministic approach," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 1872–1905, 2011.

[3] A. F. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 789–804, 2006.

[4] Y. Polyanskiy and S. Verdú, "Arimoto channel coding converse and Rényi divergence," in *Proc. of Allerton Conference*, 2010.

[5] S. L. Fong and V. Y. F. Tan, "Strong converse theorems for classes of multimessage multicast networks: A Rényi divergence approach," arXiv:1407.2417v2 [cs.IT], Jul. 2014.