SUPER: Sparse signals with Unknown Phases Efficiently Recovered

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I. Introduction

Let $A \in \mathbb{C}^{m\times n}$ be used to denote the *phase measurement matrix*, and $\mathbf{x} \in \mathbb{C}^n$ be used to denote the unknown underlying signal. Instead of *linear* measurements of the form $y = A\mathbf{x}$ as in the *compressive sensing* literature, in the *phase retrieval problem* we have m *non-linear intensity measurements* of the form $b_i = |A_i\mathbf{x}|$. Here the index i is an integer in $\{1,\ldots,m\}$ (or [m] for short), A_i is the i-th row of phase measurement matrix A and $|\cdot|$ is the absolute value.

Suppose $\mathbf{x}$ is "sparse", *i.e.*, the number of non-zero components of $\mathbf{x}$ is at most k, which is much less than the length n of $\mathbf{x}$. This assumption is not uncommon in many applications like X-ray crystallography. Then, given A and b, the goal of *compressive phrase retrieval* is to reconstruct $\mathbf{x}$ as $\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ equals $\mathbf{x}$ up to a global phase. That is, $\hat{\mathbf{x}} = \mathbf{x}e^{\iota\Theta}$ for some arbitrary fixed $\Theta \in [0, 2\pi)$. Here ι denotes the positive square root of -1. The reason we allow this degeneracy in $\hat{\mathbf{x}}$, up to a global phase factor, is that all such $\hat{\mathbf{x}}$'s result in the same measurement vector under intensity measurements. If $\hat{\mathbf{x}}$ does indeed equal $\mathbf{x}$ up to a global phase, then we denote this "equality" as $\hat{\mathbf{x}} \hat{=} \mathbf{x}$.

It is shown that $4k-1$ intensity measurements suffice to uniquely reconstruct $\mathbf{x}$ in [1] (for $\mathbf{x} \in \mathbb{R}^n$) and [2] (for $\mathbf{x} \in \mathbb{C}^n$). However, no efficient algorithm is given. The ℓ_1-regularized PhaseLift method is introduced in the compressive phase retrieval problem in [3]. In [4], it is shown that if the number of Gaussian intensity measurements is $\mathcal{O}\left(k^2 \log n\right)$, $\mathbf{x}$ can be correctly reconstructed via ℓ_1-regularized PhaseLift.

The works in [5] and the works by Jaganathan *et al.* [6], [7], [8] study the case when the phase measurement matrix is a Fourier transform matrix. [9] shows that SDP-based methods can reconstruct $\mathbf{x}$ with sparsity up to $o\left(\sqrt{n}\right)$. In [7], the algorithm based on reweighted ℓ_1-minimization with $\mathcal{O}\left(k^2 \log n\right)$ phaseless Fourier measurements is proposed to go beyond this bottleneck. When the phase measurement matrix is allowed to be designed, a matrix ensemble and a corresponding combinatorial algorithm is proposed in [7] such that $\mathbf{x}$ is correctly reconstructed with $\mathcal{O}(k \log n)$ intensity measurements in $\mathcal{O}(kn \log n)$ time. The Unicolor algorithm in [10] builds on our work [11] and is able to reconstruct a constant fraction of non-zero components of $\mathbf{x}$ with $\mathcal{O}(k)$ measurements in $\mathcal{O}(k)$ time.

To our best knowledge, in the literature, there is no construction of a measurement matrix A and a corresponding reconstruction algorithm that correctly reconstructs $\mathbf{x}$ with an order-optimal number of measurements and with near-optimal decoding complexity simultaneously.

II. Our Contribution

In this work, we focus on compressive phase retrieval problem with noiseless intensity measurements. We propose SUPER, which consists of a randomized design of the measurement matrix and a corresponding decoding algorithm that achieve the following guarantees:

Theorem 1. *(Main theorem) There exists a measurement ensemble* $\{A\}$ *and a corresponding decoding algorithm for compressive phase retrieval with the following performance:*

1) For every $\mathbf{x} \in \mathbb{C}^n$, with probability $1 - o(1)$ over the randomized design of A, the algorithm exactly reconstructs $\mathbf{x}$ up to a global phase;
2) The number of measurements $m = \mathcal{O}(k)$;
3) The decoding complexity is $\mathcal{O}(k \log k)$.

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