SUPER: Sparse signals with Unknown Phases Efficiently Recovered

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I. INTRODUCTION

Let $A \in \mathbb{C}^{m \times n}$ be used to denote the phase measurement matrix, and $x \in \mathbb{C}^n$ be used to denote the unknown underlying signal. Instead of linear measurements of the form $y = Ax$ as in the compressive sensing literature, in the phase retrieval problem we have $m$ non-linear intensity measurements of the form $b_i = |A_i x|$. Here the index $i$ is an integer in $\{1, \ldots, m\}$ (or $[m]$ for short). $A_i$ is the $i$-th row of phase measurement matrix $A$ and $|\cdot|$ is the absolute value.

Suppose $x$ is “sparse”, i.e., the number of non-zero components of $x$ is at most $k$, which is much less than the length $n$ of $x$. This assumption is not uncommon in many applications like X-ray crystallography. Then, given $A$ and $b$, the goal of compressive phase retrieval is to reconstruct $x$ as $\hat{x}$, where $\hat{x}$ equals $x$ up to a global phase. That is, $\hat{x} = xe^{i\theta}$ for some arbitrary fixed $\Theta \in [0, 2\pi)$. Here $\ell$ denotes the positive square root of $-1$. The reason we allow this degeneracy in $\hat{x}$, up to a global phase factor, is that all such $\hat{x}$’s result in the same measurement vector under intensity measurements. If $\hat{x}$ does indeed equal $x$ up to a global phase, then we denote this “equality” as $\hat{x} = x$.

It is shown that $4k - 1$ intensity measurements suffice to uniquely reconstruct $x$ in [1] (for $x \in \mathbb{R}^n$) and [2] (for $x \in \mathbb{C}^n$). However, no efficient algorithm is given. The $\ell_1$-regularized PhaseLift method is introduced in the compressive phase retrieval problem in [3]. In [4], it is shown that if the number of Gaussian intensity measurements is $O(k^2 \log n)$, $x$ can be correctly reconstructed via $\ell_1$-regularized PhaseLift.

The works in [5] and the works by Jaganathan et al. [6], [7], [8] study the case when the phase measurement matrix is a Fourier transform matrix. [9] shows that SDP-based methods can reconstruct $x$ with sparsity up to $O(\sqrt{n})$. In [7], the algorithm based on reweighted $\ell_1$-minimization with $O\left(k^2 \log n\right)$ phaseless Fourier measurements is proposed to go beyond this bottleneck. When the phase measurement matrix is allowed to be designed, a matrix ensemble and a corresponding combinatorial algorithm is proposed in [7] such that $x$ is correctly reconstructed with $O(k \log n)$ intensity measurements in $O(kn \log n)$ time. The Unicolor algorithm in [10] builds on our work [11] and is able to reconstruct a constant fraction of non-zero components of $x$ with $O(k)$ measurements in $O(k)$ time.

To our best knowledge, in the literature, there is no construction of a measurement matrix $A$ and a corresponding reconstruction algorithm that correctly reconstructs $x$ with an order-optimal number of measurements and with near-optimal decoding complexity simultaneously.

II. OUR CONTRIBUTION

In this work, we focus on compressive phase retrieval problem with noiseless intensity measurements. We propose SUPER, which consists of a randomized design of the measurement matrix and a corresponding decoding algorithm that achieve the following guarantees:

**Theorem 1. (Main theorem)** There exists a measurement ensemble $\{A\}$ and a corresponding decoding algorithm for compressive phase retrieval with the following performance:

1) For every $x \in \mathbb{C}^n$, with probability $1 - o(1)$ over the randomized design of $A$, the algorithm exactly reconstructs $x$ up to a global phase;
2) The number of measurements $m = O(k)$;
3) The decoding complexity is $O(k \log k)$.

REFERENCES